

Tensor norm and maximal singular vectors of non-negative tensors - a Perron-Frobenius theorem, a Collatz-Wielandt characterization and a generalized power method

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Abstract

We study the ℓ^{p_1, \dots, p_m} singular value problem for non-negative tensors. We prove a general Perron-Frobenius theorem for weakly irreducible and irreducible nonnegative tensors and provide a Collatz-Wielandt characterization of the maximal singular value. Additionally, we propose a higher order power method for the computation of the maximal singular vectors and show that it has an asymptotic linear convergence rate.

Keywords: Perron-Frobenius theorem for nonnegative tensors, Maximal singular value, convergence analysis of the higher order power method.

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1 Introduction

In recent years an increasing number of applications of the multilinear structure of tensors has been discovered in several disciplines, e.g. higher order statistics, signal processing, biomedical engineering, etc. [1, 2, 3]. In this paper we study the maximal singular value problem for nonnegative tensors which is induced by the variational characterization of the projective tensor norm. Let $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ and $1 < p_1, \dots, p_m < \infty$, we consider the ℓ^{p_1, \dots, p_m} singular values of f defined by Lim [4] as the critical points of the function $Q: \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m} \rightarrow \mathbb{R}$ given by

$$Q(\mathbf{x}_1, \dots, \mathbf{x}_m) := \frac{|f(\mathbf{x}_1, \dots, \mathbf{x}_m)|}{\|\mathbf{x}_1\|_{p_1} \cdot \dots \cdot \|\mathbf{x}_m\|_{p_m}},$$

where $\mathbf{x}_i \in \mathbb{R}^{d_i}$, $\|\cdot\|_p$ denotes the p -norm and

$$f(\mathbf{x}_1, \dots, \mathbf{x}_m) := \sum_{j_1 \in [d_1], \dots, j_m \in [d_m]} f_{j_1, \dots, j_m} x_{1, j_1} \cdot \dots \cdot x_{m, j_m}$$

with $[n] := \{1, \dots, n\}$. The maximum of Q is the so-called projective tensor norm [5] and we write it $\|f\|_{p_1, \dots, p_m}$. Note that the variational characterization of singular values we use here is slightly different than

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the one proposed in [4] as we have the absolute value in Q which leads to the fact that singular values of tensors are all non-negative similar to the matrix case.

The main contributions of this paper are a Perron-Frobenius Theorem for the maximal ℓ^{p_1, \dots, p_m} singular value of nonnegative tensors together with its Collatz-Wielandt characterization and a power method that computes this maximal singular value and the associated singular vectors. More precisely, let p' denote the Hölder conjugate of p and for $n \in \mathbb{N}$, let $\psi_p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $(\psi_p(\mathbf{x}))_j = |x_j|^{p-1} \text{sign}(x_j)$ for $j \in [n]$. Moreover, let $\mathbb{R}_{++}^n := \{\mathbf{x} \in \mathbb{R}^n \mid x_i > 0, i \in [n]\}$, $\mathbb{S}_{++}^d := \{(\mathbf{x}_1, \dots, \mathbf{x}_m) \mid \mathbf{x}_k \in \mathbb{R}^{d_k}, \|\mathbf{x}_k\|_{p_k} = 1 \text{ and } \mathbf{x}_k \in \mathbb{R}_{++}^{d_k}, k \in [m]\}$ and for $i \in [m], k \in [m] \setminus \{i\}, j_k \in [d_k]$ let $s_{i,k,j_k} : \mathbb{S}_{++}^d \rightarrow \mathbb{R}$ be defined by

$$s_{i,k,j_k}(\mathbf{x}) := \psi_{p'_k} \left(\sum_{j_i \in [d_i]} \psi_{p'_i} \left(\frac{\partial}{\partial x_{i,j_i}} f(\mathbf{x}) \right) \frac{\partial^2}{\partial x_{i,j_i} \partial x_{k,j_k}} f(\mathbf{x}) \right),$$

then we have the following result.

Theorem 1. Let $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ be a nonnegative weakly irreducible tensor and $1 < p_1, \dots, p_m < \infty$ such that there exists $i \in [m]$ with

$$m-1 \leq (p_i-1) \left(\min_{k \in [m] \setminus \{i\}} p_k - (m-1) \right).$$

Then there exists a unique vector $\mathbf{x}^* \in \mathbb{S}_{++}^d$ such that $Q(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$. Furthermore, it holds

$$\max_{\mathbf{x} \in \mathbb{S}_{++}^d} \prod_{\nu \in [m] \setminus \{i\}} \min_{j_\nu \in [d_\nu]} \left(\frac{s_{i,\nu,j_\nu}(\mathbf{x})}{x_{\nu,j_\nu}} \right)^{p_\nu-1} = \|f\|_{p_1, \dots, p_m}^{p'_i(m-1)} = \min_{\mathbf{x} \in \mathbb{S}_{++}^d} \prod_{\nu \in [m] \setminus \{i\}} \max_{j_\nu \in [d_\nu]} \left(\frac{s_{i,\nu,j_\nu}(\mathbf{x})}{x_{\nu,j_\nu}} \right)^{p_\nu-1},$$

and if f is irreducible then $\mathbf{x}^* > 0$ is the unique nonnegative ℓ^{p_1, \dots, p_m} singular vector of f , up to scale.

Note that our theory can be extended so that the maximum (respectively the minimum) in the min-max characterization of Theorem 1 is taken over $\{(\mathbf{x}_1, \dots, \mathbf{x}_m) \mid \mathbf{x}_k \in \mathbb{R}_{++}^{d_k}, k \in [m]\}$ instead of \mathbb{S}_{++}^d . A key element of our proof is the construction of a bijection between the ℓ^{p_1, \dots, p_m} -singular vectors of f and the points $\mathbf{x} \in \mathbb{S}_{++}^{d-d_i} := \{(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_m) \mid \mathbf{x}_k \in \mathbb{R}_{++}^{d_k} \text{ and } \|\mathbf{x}_k\|_{p_k} = 1, k \in [m] \setminus \{i\}\}$ satisfying $s_{i,k,j_k}(\mathbf{x}) = \lambda^{p'_i(p'_k-1)} x_{k,j_k}$ for every $k \in [m] \setminus \{i\}$ and $j_k \in [d_k]$ (see Proposition 5). Based on this observation we also build an algorithm that computes the maximal singular vectors of f , which is in the next theorem.

Theorem 2. Let $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ be a nonnegative weakly irreducible tensor and $1 < p_1, \dots, p_m < \infty$ satisfying the assumption of Theorem 1. Let $(\lambda_-^k)_{k \in \mathbb{N}}, (\lambda_+^k)_{k \in \mathbb{N}}$ and $(\mathbf{x}^k)_{k \in \mathbb{N}}$ be the sequences produced by the Higher-order Generalized Power Method (see p.14). Moreover, let $\tilde{\mathbf{x}}^* \in \mathbb{S}_{++}^d$ be a singular vector of f satisfying $Q(\tilde{\mathbf{x}}^*) = \|f\|_{p_1, \dots, p_m}$ and let $(\|f\|_{p_1, \dots, p_m}, \mathbf{x}^*) = \Phi_i^{-1}(\|f\|_{p_1, \dots, p_m}, \tilde{\mathbf{x}}^*)$ where Φ_i is the bijection given in Proposition 5. Then

$$\forall k \in \mathbb{N}, \quad \lambda_-^k \leq \lambda_-^{k+1} \leq \|f\|_{p_1, \dots, p_m} \leq \lambda_+^{k+1} \leq \lambda_+^k, \quad \lim_{k \rightarrow \infty} \lambda_-^k = \|f\|_{p_1, \dots, p_m} = \lim_{k \rightarrow \infty} \lambda_+^k, \quad \lim_{k \rightarrow \infty} \mathbf{x}^k = \mathbf{x}^*$$

and there exists $0 < \nu < 1$, $k_0 \in \mathbb{N}$ and a norm $\|\cdot\|_G$ such that

$$\|\mathbf{x}^{k+1} - \mathbf{x}^*\|_G \leq \nu \|\mathbf{x}^k - \mathbf{x}^*\|_G \quad \forall k \geq k_0.$$

Note that the computation of matrix norms [6] and tensor norms [7] is NP-hard in general and thus the restriction to nonnegative matrices resp. tensors is crucial. In the case of matrices ($m = 2$) a power method for the computation of a general (p_1, p_2) -norm of a nonnegative matrix and its associated singular vectors has been considered by Boyd [8] already in 1974. Recently, the paper has been reconsidered by [9] where uniqueness of the strictly positive singular vectors is shown under the condition that the matrix is strictly positive. Our Perron-Frobenius theorem extends this uniqueness result to irreducible matrices (note that the notion of weakly irreducible and irreducible coincide for matrices and reduce to the standard notion of irreducibility).

On the tensor side ($m > 2$) Perron-Frobenius Theorems for nonnegative tensors have already been established for H -eigenvalues [10] ($p_i = m, i = 1, \dots, m$) and general ℓ^{p_1, \dots, p_m} singular vectors [11] for $p_i \geq m, i = 1, \dots, m$. Our Perron-Frobenius theorem in Theorem 1 extends the range of (p_1, \dots, p_m) to

$$m - 1 \leq (p_i - 1) \left(\min_{k \in [m] \setminus \{i\}} p_k - (m - 1) \right).$$

In particular, this allows to have one p_i to be arbitrarily close to 1 given that the other p_k are sufficiently large or alternatively, all except one p_i can be arbitrarily close to $m - 1$ whereas p_i has to be sufficiently large. Moreover, our Collatz-Wielandt characterization seems to be the first one for the case where the p_i are not all equal. Our result also leads to a slight generalization for the singular vectors of a partially symmetric nonnegative tensor introduced in [12] which is discussed in more detail in Section 5.

The power method for tensors was first introduced by Ng, Qi and Zhou in [13] for the computation of the maximal H -eigenvalue of an irreducible nonnegative tensor. It was generalized in [11] where the method can also be used for computing ℓ^{p_1, \dots, p_m} singular vectors for weakly primitive nonnegative tensors. However, their method applies only in the case when $p_1 = \dots = p_m$ while our higher order power method needs only weak irreducibility of the nonnegative tensor and does not require p_1, \dots, p_m to be equal.

Let us describe the organization of this paper. In Section 2, we prove some general properties of the ℓ^{p_1, \dots, p_m} singular values of f . In Section 3, we restrict our study to nonnegative tensors and provide criteria to guarantee the existence of some strictly positive singular vector¹ of f . In Section 4, we provide our Perron-Frobenius Theorem and Collatz-Wielandt characterization of the maximal singular value. We discuss the relation between ℓ^{p_1, \dots, p_m} -singular value and other spectral problems for tensors in Section 5. The higher order power method together with its convergence rate are introduced in Section 6. Finally, in Section 7 we do a small numerical experiment and compare our power method to the one proposed in [11].

¹For simplicity we speak in the following of the singular vector even though this corresponds to a set of m singular vectors of a m -th order tensor.

2 Notations and characterization of the singular spectrum

For $f \in \mathbb{R}^{d_1, \dots, d_m}$, $f \geq 0$ (resp. $f > 0$) mean that every entry of f is nonnegative (resp. strictly positive), furthermore we write $f \leq g$ (resp. $f < g$) if $g - f \geq 0$ (resp. $g - f > 0$). Let $\mathfrak{R}^d := \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_m}$, we use bold letters without index to denote vectors in \mathfrak{R}^d , bold letters with index $i \in [m]$ denote vectors in \mathbb{R}^{d_i} and the components of these vectors are written in normal font, i.e. $\mathbf{x} \in \mathfrak{R}^d$, $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_m)$, $\mathbf{x}_i \in \mathbb{R}^{d_i}$, $\mathbf{x}_i = (x_{i,1}, \dots, x_{i,d_i})$, $x_{i,j_i} \in \mathbb{R}$. We denote by \mathbb{S}^d the “unit sphere” in \mathfrak{R}^d , i.e. $\mathbf{x} \in \mathbb{S}^d$ if and only if $\|\mathbf{x}_i\|_{p_i} = 1$ for every $i \in [m]$. We write p' to denote the Hölder conjugate of $1 < p < \infty$ (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$). For $i \in [m]$, set $\mathfrak{R}^{d-d_i} := \mathbb{R}^{d_1} \times \dots \times \mathbb{R}^{d_{i-1}} \times \mathbb{R}^{d_{i+1}} \times \dots \times \mathbb{R}^{d_m}$ and let \mathbb{S}^{d-d_i} be the set of $\mathbf{x} \in \mathfrak{R}^{d-d_i}$ such that $\|\mathbf{x}_k\|_{p_k} = 1$ for every $k \in [m] \setminus \{i\}$. Furthermore, for $n \in \mathbb{N}$ and $V \in \{\mathfrak{R}^d, \mathbb{S}^d, \mathfrak{R}^{d-d_i}, \mathbb{S}^{d-d_i}, \mathbb{R}^n\}$ we write V_+ (resp. V_{++}) the restriction of V to the positive cone (resp. to the interior of the positive cone), e.g. $\mathfrak{R}_+^d := \{\mathbf{x} \in \mathfrak{R}^d \mid \mathbf{x} \geq 0\}$ and $\mathbb{S}_{++}^{d-d_i} := \{\mathbf{x} \in \mathbb{S}^{d-d_i} \mid \mathbf{x} > 0\}$. We follow a similar system of notation as vectors for the gradient of f , that is $\nabla f(\mathbf{x}) \in \mathfrak{R}^d$, $\nabla_i f(\mathbf{x}) \in \mathbb{R}^{d_i}$, $\partial_{i,j_i} f(\mathbf{x}) \in \mathbb{R}$, $\nabla f(\mathbf{x}) = (\nabla_1 f(\mathbf{x}), \dots, \nabla_m f(\mathbf{x}))$ and $\nabla_i f(\mathbf{x}) = (\partial_{i,1} f(\mathbf{x}), \dots, \partial_{i,d_i} f(\mathbf{x}))$ where $\partial_{i,j_i} := \frac{\partial}{\partial x_{i,j_i}}$. In particular, note that $f(\mathbf{x}) = \langle \nabla_i f(\mathbf{x}), \mathbf{x}_i \rangle$ and $\partial_{i,j_i} f(\mathbf{x}) = f(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{e}_{(i,j_i)}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_m)$, where $\mathbf{e}_{(i,1)}, \dots, \mathbf{e}_{(i,d_i)}$ is the canonical basis of \mathbb{R}^{d_i} . Furthermore, for $g \in \mathbb{R}^{d_1 \times \dots \times d_m}$ we denote by $|g| \in \mathbb{R}^{d_1 \times \dots \times d_m}$ the tensor such that $|g|_{j_1, \dots, j_m} = |g_{j_1, \dots, j_m}|$ for every $j_1 \in [d_1], \dots, j_m \in [d_m]$.

For $1 < q < \infty$ define $\psi_q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $(\psi_q(\mathbf{y}))_j = |y_j|^{q-1} \text{sign}(y_j)$ for every $1 \leq j \leq n$. Note that $\psi_q(\psi_{q'}(\mathbf{x})) = \mathbf{x}$, $\|\psi_q(\mathbf{x})\|_r = \|\mathbf{x}\|_{r(q-1)}^{q-1}$ and $\nabla \|\mathbf{y}\|_q = \|\mathbf{y}\|_q^{1-q} \psi_q(\mathbf{y})$. Consider the function $S : \mathfrak{R}^d \rightarrow \mathbb{R}$ defined by $S(\mathbf{x}) = \|\mathbf{x}_1\|_{p_1} \cdot \dots \cdot \|\mathbf{x}_m\|_{p_m}$.

One can conclude that a critical point \mathbf{x} of Q such that $f(\mathbf{x}) \neq 0$ (Q is not differentiable at \mathbf{x} if $f(\mathbf{x}) = 0$) must satisfy the following nonlinear system of equations

$$\text{sign}(f(\mathbf{x})) \nabla_i f(\mathbf{x}) = Q(\mathbf{x}) S(\mathbf{x}) \|\mathbf{x}_i\|_{p_i}^{-p_i} \psi_{p_i}(\mathbf{x}_i) \quad \forall i \in [m]. \quad (1)$$

Now, for $i \in [m]$ consider

$$\sigma_i : \mathfrak{R}^d \rightarrow \mathbb{R}^{d_i}, \quad \mathbf{x} \mapsto \text{sign}(f(\mathbf{x})) \psi_{p_i'}(\nabla_i f(\mathbf{x})), \quad (2)$$

then $\mathbf{x} \in \mathbb{S}^d$ is a critical point of Q if and only if $f(\mathbf{x}) \neq 0$ and $\sigma_i(\mathbf{x}) = Q(\mathbf{x})^{p_i'-1} \mathbf{x}_i$ for $i \in [m]$. This motivates our choice for the definition of $\mathfrak{G}(f)$, the set of critical values of the function Q , as follow:

$$\mathfrak{G}(f) := \left\{ \lambda \in \mathbb{R} \setminus \{0\} \mid \exists \mathbf{x} \in \mathbb{S}^d \text{ with } \sigma_i(\mathbf{x}) = \lambda^{p_i'-1} \mathbf{x}_i \quad \forall i \in [m] \right\}.$$

The proof of Proposition 3 resembles the study of Z -eigenvalues in [14].

Proposition 3. Let $1 < p_1, \dots, p_m < \infty$ and $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$. If $f \neq 0$, then $\mathfrak{G}(f) \neq \emptyset$ and for every $\lambda \in \mathfrak{G}(f)$ we have $0 < \lambda \leq \min_{i \in [m]} \max_{l_i \in [d_i]} d_i^{1/p_i'} \partial_{i,l_i} \tilde{f}(\mathbf{e})$, where $\mathbf{e} = (1, 1, \dots, 1) \in \mathfrak{R}^d$ and $\tilde{f} := |f|$.

Proof. First of all, note that for every $\mathbf{x} \in \mathfrak{R}^d$ such that $S(\mathbf{x}) > 0$ we have $Q(\mathbf{x}) = Q\left(\frac{\mathbf{x}_1}{\|\mathbf{x}\|_{p_1}}, \dots, \frac{\mathbf{x}_m}{\|\mathbf{x}\|_{p_m}}\right)$. Thus, $\|f\|_{p_1, \dots, p_m} = \sup_{\mathbf{x} \in \mathbb{S}^d} Q(\mathbf{x})$ and since $\mathbf{x} \mapsto Q(\mathbf{x})$ is a continuous function on the compact set \mathbb{S}^d , it

reaches its maximum. It follows that $\mathfrak{G}(f) \neq \emptyset$. Let $\mathbf{x}^* \in \mathbb{S}^d$ be a critical point of Q associated to $\lambda \in \mathfrak{G}(f)$. Let $i \in [m]$, then, from $\|\mathbf{x}_i^*\|_{p_i} = 1$, it follows that there exists some $j_i \in [d_i]$ such that $|x_{i,j_i}^*| \geq d_i^{-1/p_i}$. In particular, the fact that ψ_{p_i} is an increasing function implies $|\psi_{p_i}(x_{i,j_i}^*)| = \psi_{p_i}(|x_{i,j_i}^*|) \geq \psi_{p_i}(d_i^{-1/p_i})$. Since $\mathbf{x}^* \in \mathbb{S}^d$ is a critical point of Q , we have $|\lambda \psi_{p_i}(x_{i,j_i}^*)| = |\partial_{i,j_i} f(\mathbf{x}^*)|$. Thus,

$$\lambda = \lambda d_i^{1/p_i' - 1/p_i} = \lambda d_i^{1/p_i'} \psi_{p_i}(d_i^{-1/p_i}) \leq d_i^{1/p_i'} |\lambda \psi_{p_i}(x_{i,j_i}^*)| = d_i^{1/p_i'} |\partial_{i,j_i} f(\mathbf{x}^*)| \leq d_i^{1/p_i'} \max_{l_i \in [d_i]} |\partial_{i,l_i} f(\mathbf{x}^*)|.$$

Since $\|\mathbf{x}_k^*\|_{p_k} = 1$, we must have $|x_{k,j_k}^*| \leq 1$ for every $k \in [m]$ and $j_k \in [d_k]$, i.e. $|\mathbf{x}^*| \leq \mathbf{e}$, where $\mathbf{e} \in \mathfrak{R}^d$ is the vector whose entries are all 1. Hence, for every $l_i \in [d_i]$, it holds $|\partial_{i,l_i} f(\mathbf{x}^*)| \leq \partial_{i,l_i} \tilde{f}(|\mathbf{x}^*|) \leq \partial_{i,l_i} \tilde{f}(\mathbf{e})$. Taking the minimum over all $i \in [m]$ concludes the proof. \square

Note that for every $f \neq 0$ and every $1 < p_1, \dots, p_m < \infty$, there exists always at least one singular value.

Example 4. Let $A \in \mathbb{R}^{2 \times 2}$ be the matrix defined by $A_{1,1} = A_{2,2} = 0$ and $A_{1,2} = -A_{2,1} = 1$. For every $1 < p_1, p_2 < \infty$, the vector $\mathbf{x}^* = \left(2^{-\frac{1}{p_1}}, -2^{-\frac{1}{p_1}}, 2^{-\frac{1}{p_2}}, 2^{-\frac{1}{p_2}}\right) \in \mathbb{R}^2 \times \mathbb{R}^2$ is a ℓ^{p_1, p_2} singular vector of A associated to the singular value $\lambda = 2^{1 - \frac{1}{p_1} - \frac{1}{p_2}}$.

We formulate now an equivalent characterization of $\mathfrak{G}(f)$. Note that for every $\mathbf{z}_i, \mathbf{y}_i \in \mathbb{R}^{d_i}$ we have $\nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{y}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_m) = \nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{z}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_m)$, thus we may, without ambiguities, abuse notation and write $\nabla_i f(\mathbf{x})$ regardless if $\mathbf{x} \in \mathfrak{R}^d$ or $\mathbf{x} \in \mathfrak{R}^{d-d_i}$. For $i \in [m]$ and $\mathbf{x} \in \mathfrak{R}^{d-d_i}$, let S_i and Q_i be the functions defined by $S_i(\mathbf{x}) := \|\mathbf{x}_1\|_{p_1} \cdots \|\mathbf{x}_{i-1}\|_{p_{i-1}} \|\mathbf{x}_{i+1}\|_{p_{i+1}} \cdots \|\mathbf{x}_m\|_{p_m}$ and

$$Q_i: \{\mathbf{x} \in \mathfrak{R}^{d-d_i} \mid S_i(\mathbf{x}) > 0\} \rightarrow \mathbb{R}, \quad \mathbf{x} \mapsto \frac{\|\nabla_i f(\mathbf{x})\|_{p_i'}}{S_i(\mathbf{x})}.$$

Observe that $\mathbf{x} \in \mathfrak{R}^{d-d_i}$ such that $S_i(\mathbf{x}) \neq 0$ and $\nabla_i f(\mathbf{x}) \neq 0$ is a critical point of Q_i if and only if it satisfies

$$S_i(\mathbf{x})^{p_i'} \|\mathbf{x}_k\|_{p_k}^{p_k} \left\langle \psi_{p_i'}(\nabla_i f(\mathbf{x})), \nabla_i \partial_{k,j_k} f(\mathbf{x}) \right\rangle = Q_i(\mathbf{x})^{p_i'} \psi_{p_k}(x_{k,j_k}), \quad \forall k \in [m] \setminus \{i\}, j_k \in [d_k]. \quad (3)$$

Here, we used $\partial_{i,j_i} \partial_{k,j_k} f(\mathbf{x}) = \partial_{k,j_k} \partial_{i,j_i} f(\mathbf{x})$ as $f \in C^2(\mathfrak{R}^d)$. It follows that every $\mathbf{x} \in \mathbb{S}^{d-d_i}$ such that $\nabla_i f(\mathbf{x}) \neq 0$ is a critical point of Q_i if and only if $s_{i,k}(\mathbf{x}) = Q_i(\mathbf{x})^{p_i'(p_k'-1)} \mathbf{x}_k$ for every $k \in [m] \setminus \{i\}$, where

$$s_{i,k}: \mathfrak{R}^{d-d_i} \rightarrow \mathbb{R}^{d_k}, \quad \mathbf{x} \mapsto \psi_{p_k'} \left(\nabla_k f \left(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \psi_{p_i'}(\nabla_i f(\mathbf{x})), \mathbf{x}_{i+1}, \dots, \mathbf{x}_m \right) \right). \quad (4)$$

In the next proposition we show equivalence between the critical points of Q_i and those of Q .

Proposition 5. Let $1 < p_1, \dots, p_m < \infty$ and $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ with $f \neq 0$, denote by $C^* \subset \mathfrak{G}(f) \times \mathbb{S}^d$ the set of pairs (λ, \mathbf{x}^*) where \mathbf{x}^* is a critical point of Q associated to the critical value λ . Furthermore, for $i \in [m]$, let $\mathfrak{G}_i(f) := \{\lambda \in \mathbb{R} \setminus \{0\} \mid \exists \mathbf{x} \in \mathbb{S}^{d-d_i} \text{ with } s_{i,k}(\mathbf{x}) = \lambda^{p_i'(p_k'-1)} \mathbf{x}_k \forall k \in [m] \setminus \{i\}\}$ and $C_i^* \subset \mathfrak{G}_i(f) \times \mathbb{S}^{d-d_i}$ be the set of all pairs (λ, \mathbf{x}^*) where \mathbf{x}^* is a critical point of Q_i associated to the critical value λ . Then $\mathfrak{G}(f) = \mathfrak{G}_i(f)$ and, with $\varsigma_i(\mathbf{x}) := \text{sign} \left(f \left(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \psi_{p_i'}(\lambda^{-1} \nabla_i f(\mathbf{x})), \mathbf{x}_{i+1}, \dots, \mathbf{x}_m \right) \right)$, the function

$$\Phi_i: C_i^* \rightarrow C^*, \quad (\lambda, \mathbf{x}) \mapsto \left(\lambda, \left(\mathbf{x}, \dots, \mathbf{x}_{i-1}, \psi_{p_i'}(\varsigma_i(\mathbf{x}) \lambda^{-1} \nabla_i f(\mathbf{x})), \mathbf{x}_{i+1}, \dots, \mathbf{x}_m \right) \right)$$

is a bijection.

Proof. First, we show that Φ_i is well defined. Let $(\lambda, \mathbf{x}) \in C_i^*$ and $(\lambda, \mathbf{x}^*) := \Phi_i(\lambda, \mathbf{x})$. Using Equation (3), $\nabla_i f(\mathbf{x}) = \nabla_i f(\mathbf{x}^*)$, $\mathbf{x}_i^* = \psi_{p_i'}(\varsigma_i(\mathbf{x}) \lambda^{-1} \nabla_i f(\mathbf{x}))$ and $\nabla_i \partial_{k,j_k} f(\mathbf{x}) = \nabla_i \partial_{k,j_k} f(\mathbf{x}^*)$, one can show that $\lambda \psi_{p_k}(\mathbf{x}_k^*) = \varsigma_i(\mathbf{x}) \nabla_k f(\mathbf{x}^*)$ for every $k \in [m]$, $j_k \in [d_k]$. Furthermore, $\text{sign}(f(\mathbf{x}^*)) = \text{sign}(\langle \nabla_i f(\mathbf{x}^*), \mathbf{x}_i^* \rangle) = \varsigma_i(\mathbf{x}) \text{sign}(\lambda^{1-p_i'} \|\nabla_i f(\mathbf{x})\|_{p_i'}^{p_i'}) = \varsigma_i(\mathbf{x})$ and $\|\mathbf{x}_i^*\|_{p_i} = 1$ because $\|\nabla_i f(\mathbf{x})\|_{p_i'}^{p_i'-1} = Q_i(\mathbf{x})^{p_i'-1} = \lambda^{p_i'-1}$, so $(\lambda, \mathbf{x}^*) \in C^*$. Injectivity is straightforward by definition of Φ_i and surjectivity is shown by noticing that if $(\lambda, \mathbf{x}^*) \in C^*$ then $(\lambda, \mathbf{x}^*) = \Phi_i(\lambda, (\mathbf{x}_1^*, \dots, \mathbf{x}_{i-1}^*, \mathbf{x}_{i+1}^*, \dots, \mathbf{x}_m^*))$. Finally, the fact that Φ_i is a bijection implies $\mathfrak{G}(f) = \mathfrak{G}_i(f)$. \square

The previous proposition implies that the maximum value of Q_i is $\|f\|_{p_1, \dots, p_m}$. Moreover, if we know a maximizer of Q_i then we can construct a maximizer of Q and vice-versa. This result can be seen as a generalization of the variational characterization of the singular values and singular vectors of matrix (see e.g. Theorem 8.3-1 [15]).

3 Nonnegative tensors and positive singular vectors

Now, we focus our study on tensors $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ with nonnegative coefficients, i.e. $f \geq 0$.

Lemma 6. Let $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ and $\mathbf{x}, \mathbf{y} \in \mathfrak{R}_+^d$ such that $f \geq 0$ and $0 \leq \mathbf{x} \leq \mathbf{y}$, then $0 \leq f(\mathbf{x}) \leq f(\mathbf{y})$, $0 \leq \nabla_k f(\mathbf{x}) \leq \nabla_k f(\mathbf{y})$, $0 \leq \sigma_k(\mathbf{x}) \leq \sigma_k(\mathbf{y})$ and $0 \leq s_{i,k}(\mathbf{x}) \leq s_{i,k}(\mathbf{y})$ for every $k, i \in [m]$ with $k \neq i$.

Proof. Straightforward computation. \square

We recall the definition of irreducible and weakly irreducible tensors.

Definition 7 ([11]). Let $f \geq 0$ and $G(f) = (V, E(f))$ an undirected m -partite graph with $V := (\{1\} \times [d_1]) \cup \dots \cup (\{m\} \times [d_m])$ and such that for every $k, l \in [m]$ with $k \neq l$, we have $((k, j_k), (l, j_l)) \in E(f)$ if and only if there exist $j_\nu \in [d_\nu]$ for $\nu \in [m] \setminus \{k, l\}$ with $f_{j_1, \dots, j_m} > 0$.

- i) We say that f is irreducible if for each proper nonempty subset $\emptyset \neq J \subsetneq V$ the following holds: Let $I = V \setminus J$, then there exist $(k, j_k) \in J$ and $(l, j_l) \in I$ for $l \in [m] \setminus \{k\}$ such that $f_{j_1, \dots, j_m} > 0$.
- ii) We say that f is weakly irreducible if $G(f)$ is connected.

The next proposition lists some useful properties of nonnegative tensors.

Proposition 8. For $1 < p_1, \dots, p_m < \infty$, $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ and $\boldsymbol{\alpha} := (\alpha_0, \alpha_1, \dots, \alpha_m) > 0$, consider the function $T_{\boldsymbol{\alpha}}: \mathfrak{R}^d \rightarrow \mathfrak{R}^d$ defined by $T_{\boldsymbol{\alpha}}(\mathbf{z}) := \alpha_0 \mathbf{z} + (\alpha_1 \sigma_1(\mathbf{z}), \dots, \alpha_m \sigma_m(\mathbf{z}))$.

- a) $f \geq 0$ if and only if $\alpha_0 \mathbf{z} \leq T_{\boldsymbol{\alpha}}(\mathbf{z})$ for every $\mathbf{z} \in \mathbb{S}_+^d$.
- b) If $f \geq 0$ is weakly irreducible, then $\alpha_0 \mathbf{z} < T_{\boldsymbol{\alpha}}(\mathbf{z})$ for every $\mathbf{z} \in \mathbb{S}_{++}^d$.
- c) $f \geq 0$ is irreducible if and only if there is some $n \in \mathbb{N}$ such that $T_{\boldsymbol{\alpha}}^n(\mathbf{z}) > 0$ for every $\mathbf{z} \in \mathbb{S}_+^d$, where $T_{\boldsymbol{\alpha}}^n(\mathbf{x})$ is recursively defined by $T_{\boldsymbol{\alpha}}^n(\mathbf{x}) = T_{\boldsymbol{\alpha}}^{n-1}(T_{\boldsymbol{\alpha}}(\mathbf{x}))$.
- d) $f > 0$ if and only if $T_{\boldsymbol{\alpha}}(\mathbf{z}) > 0$ for every $\mathbf{z} \in \mathbb{S}_+^d$.

Proof. Let us recall that $\mathbf{z} \in \mathbb{S}^d$ implies $\mathbf{z}_k \neq 0$ for every $k \in [m]$.

- a) If $f \geq 0$ and $\mathbf{z} \in \mathbb{S}_+^d$, then, by Lemma 6, $\sigma_i(\mathbf{z}) \geq 0$ for any $i \in [m]$ and thus $\alpha_0 \mathbf{z} \leq T_{\boldsymbol{\alpha}}(\mathbf{z})$. Now, suppose that there is $\mathbf{z} \in \mathbb{S}_+^d$, $i \in [m]$ and $j_i \in [d_i]$ such that $(T_{\boldsymbol{\alpha}}(\mathbf{z}))_{i,j_i} < \alpha_0 z_{i,j_i}$. It follows that $0 > (T_{\boldsymbol{\alpha}}(\mathbf{z}))_{i,j_i} - \alpha_0 z_{i,j_i} = \alpha_i \sigma_{i,j_i}(\mathbf{z}) + \alpha_0 z_{i,j_i} - \alpha_0 z_{i,j_i} = \alpha_i \sigma_{i,j_i}(\mathbf{z})$. Since $\alpha_i > 0$, this is possible if and only if $\partial_{i,j_i} f(\mathbf{z}) < 0$. However, $\mathbf{z} \geq 0$, so there must be some entry of f which is strictly negative.

- b) Let $f \geq 0$ be weakly irreducible and $\mathbf{z} \in \mathbb{S}_{++}^d$. We need to show that $\sigma_{k,j_k}(\mathbf{z}) > 0$ for every $k \in [m]$ and $j_k \in [d_k]$. Suppose by contradiction that there is some $i \in [m]$ and $j_i \in [d_i]$ such that $\partial_{i,j_i} f(\mathbf{z}) = 0$. Then, from $\mathbf{z} > 0$ and $f \geq 0$ follows that $f_{j_1, \dots, j_m} = 0$ for every $j_k \in [d_k]$ and $k \in [m] \setminus \{i\}$, thus the vertex (i, j_i) is not connected to any other vertex in the graph $G(f)$ associated to f . This is a contradiction to the fact that f is weakly irreducible.
- c) Suppose that $f \geq 0$ is irreducible and for $\mathbf{z} \in \mathbb{S}_+^d$ let $J_{\mathbf{z}_i} := \{j_i \in [d_i] \mid z_{i,j_i} = 0\}$ and $J_{\mathbf{z}} := J_{\mathbf{z}_1} \times \dots \times J_{\mathbf{z}_m}$. Note that $\mathbf{z} \in \mathbb{S}_+^d$ implies $J_{\mathbf{z}_i} \neq [d_i]$ for every $i \in [m]$. $J_{T_{\alpha}(\mathbf{z})} \subset J_{\mathbf{z}}$ follows from $f \geq 0$ and the property discussed above. We show that if $J_{\mathbf{z}} \neq \emptyset$, then $J_{T_{\alpha}(\mathbf{z})}$ is strictly contained in $J_{\mathbf{z}}$. So, suppose $J_{\mathbf{z}} \neq \emptyset$ and assume by contradiction that $J_{T_{\alpha}(\mathbf{z})} = J_{\mathbf{z}}$. Let $\nu \in [m]$ and $j_{\nu} \in J_{\mathbf{z}_{\nu}}$, then $0 = \alpha_0 z_{\nu, j_{\nu}} + \alpha_{\nu} \sigma_{\nu, j_{\nu}}(\mathbf{z}) = \alpha_{\nu} \psi_{p_{\nu}}(\partial_{z_{\nu, j_{\nu}}} f(\mathbf{z}))$ and

$$\sum_{\substack{j_1 \in [d_1] \setminus J_{\mathbf{z}_1}, \dots, j_{\nu-1} \in [d_{\nu-1}] \setminus J_{\mathbf{z}_{\nu-1}}, \\ j_{\nu+1} \in [d_{\nu+1}] \setminus J_{\mathbf{z}_{\nu+1}}, \dots, j_m \in [d_m] \setminus J_{\mathbf{z}_m}}} f_{j_1, \dots, j_m} \underbrace{z_{1,j_1} \cdots z_{\nu-1, j_{\nu-1}} z_{\nu+1, j_{\nu+1}} \cdots z_{m, j_m}}_{>0} = 0$$

This implies that for all $\nu \in [m], j_{\nu} \in J_{\mathbf{z}_{\nu}}, k \in [m] \setminus \{\nu\}$ and $j_k \in [d_k] \setminus J_{\mathbf{z}_k}$, we have $f_{j_1, \dots, j_m} = 0$, a contradiction to the irreducibility of f . Thus $J_{T_{\alpha}(\mathbf{z})}$ is strictly contained in $J_{\mathbf{z}}$. Using the fact that $\mathbf{z} \in \mathbb{S}^d$ has at least m nonzero components (because $\|\mathbf{z}_i\|_{p_i} = 1$ for every $i \in [m]$), we get the existence of $n \in \mathbb{N}$ such that $T_{\alpha}^n(\mathbf{z}) > 0$. Now, assume that there is $\mathbf{z} \in \mathbb{S}_+^d, i \in [m]$ and $j_i \in [d_i]$ such that $(T_{\alpha}^k(\mathbf{z}))_{i, j_i} = 0$ for every $k \in \mathbb{N}$. Suppose by contradiction that f is irreducible. Since $f \geq 0$, we must have $z_{i, j_i} = 0$. Thus, if $\kappa(\mathbf{z}) \in \mathbb{N}$ denotes the cardinality of $J_{\mathbf{z}}$, we have $\kappa(\mathbf{z}) > 0$. But we assumed f to be irreducible and by the same arguments as above, for every $\mathbf{x} \in \mathbb{S}_+^d$, if $\kappa(\mathbf{x}) > 0$, then $\kappa(\mathbf{x}) > \kappa(T_{\alpha}(\mathbf{x}))$. A contradiction to $\kappa(T_{\alpha}^k(\mathbf{z})) > 0$ for every $k \in \mathbb{N}$.

- d) Suppose $f > 0$ and let $i \in [m], j_i \in [d_i]$ and $\mathbf{z} \in \mathbb{S}_+^d$. Since $\mathbf{z} \in \mathbb{S}_+^d$, for every $k \in [m]$ there exists some $j_k \in [d_k]$ such that $z_{k, j_k} > 0$. It follows that

$$0 < f_{j_1, \dots, j_m} z_{1, j_1} \cdots z_{i-1, j_{i-1}} z_{i+1, j_{i+1}} \cdots z_{m, j_m} \leq \partial_{i, j_i} f(\mathbf{z}) \quad \forall j_i \in [d_i]$$

and thus $\sigma_{i, j_i}(\mathbf{z}) > 0$. This is true for every $i \in [m]$ and $j_i \in [d_i]$, thus $0 < (\alpha_1 \sigma_1(\mathbf{z}), \dots, \alpha_m \sigma_m(\mathbf{z})) \leq T_{\alpha}(\mathbf{z})$. Now, suppose that there exist $l_1 \in [d_1], \dots, l_m \in [d_m]$ such that $f_{l_1, \dots, l_m} \leq 0$. Consider the vector \mathbf{z} defined for every $k \in [m]$ and $j_k \in [d_k]$ by $z_{k, j_k} = (d_1 - 1)^{-1/p_1}$ if $k = 1$ and $j_1 \neq l_1$, $z_{k, j_k} = 1$ if $k > 1$ and $j_k = l_k$, $z_{k, j_k} = 0$ else. Then $\mathbf{z} \in \mathbb{S}_+^d, z_{1, l_1} = 0$ and $\partial_{1, l_1} f(\mathbf{z}) = f_{l_1, \dots, l_m} \leq 0$. It follows that $(T_{\alpha}(\mathbf{z}))_{1, j_1} \leq 0$. \square

Note that it is proved in Lemma 3.1, [11], that every irreducible tensor is weakly irreducible. The next example shows that the reverse implication of Proposition 8, b) is not true in general.

Example 9. Let $f \in \mathbb{R}^{2 \times 2 \times 2}$ be defined by $f_{1,1,1} = f_{2,2,2} = 1$ and zero else. Then, for every $\mathbf{x} > 0$, we have $\nabla f(\mathbf{x}) = (\mathbf{x}_2 \circ \mathbf{x}_3, \mathbf{x}_1 \circ \mathbf{x}_3, \mathbf{x}_1 \circ \mathbf{x}_2)$, where \circ denotes the Hadamard product. If T_{α} is defined as in Proposition 8, then $T_{\alpha}(\mathbf{x}) > \alpha_0 \mathbf{x}$ for every $\mathbf{x} > 0$, however f is not weakly irreducible.

Corollary 10. Let $f \geq 0$ be a weakly irreducible tensor and $1 < p_1, \dots, p_m < \infty$, then for every $\mathbf{z} \in \mathfrak{R}_{++}^{d-d_i}$ and $i, k \in [m]$ with $i \neq k$, it holds $s_{i,k}(\mathbf{z}) > 0$.

Proof. Let $f \geq 0$ and $\mathbf{z} > 0$, then $s_{i,k}(\mathbf{z}) = \sigma_k(\mathbf{z}_1, \dots, \mathbf{z}_{i-1}, \sigma_i(\mathbf{z}), \mathbf{z}_{i+1}, \dots, \mathbf{z}_m)$. It follows by Proposition 8, b) that $\sigma_i(\mathbf{z}) > 0$ and thus also $s_{i,k}(\mathbf{z}) > 0$. \square

Corollary 11. Let $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ with $f \neq 0$, $1 < p_1, \dots, p_m < \infty, (\lambda, \mathbf{x}^*) \in C_i^*, (\mu, \mathbf{y}^*) \in C^*$ and $\Phi_i : C_i^* \rightarrow C^*$ as defined in Proposition 5.

- i) If $(\lambda, \mathbf{x}^*) \geq 0$ and $f \geq 0$, then $\Phi_i(\lambda, \mathbf{x}^*) \geq 0$.
- ii) If $(\lambda, \mathbf{x}^*) > 0$ and $f \geq 0$ is weakly irreducible, then $\Phi_i(\lambda, \mathbf{x}^*) > 0$.

iii) If $(\mu, \mathbf{y}^*) \geq 0$, then $\Phi_i^{-1}(\mu, \mathbf{y}^*) \geq 0$.

iv) If $(\mu, \mathbf{y}^*) > 0$, then $\Phi_i^{-1}(\mu, \mathbf{y}^*) > 0$.

Proof. i) follows from Proposition 8, a). ii) follows from Proposition 8, b). Finally iii) and iv) follow from the definition of Φ_i . \square

As we need the existence of some $\mathbf{x}^* > 0$ such that $Q(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$ for the Collatz-Wielandt analysis, we provide here a few conditions on f and $1 < p_1, \dots, p_m < \infty$ in order to guarantee it.

Lemma 12. Let $f \geq 0$ and $1 < p_1, \dots, p_m < \infty$, then there exists some singular vector $\mathbf{x}^* \in \mathfrak{R}_+^d$ of f such that $\|f\|_{p_1, \dots, p_m} = Q(\mathbf{x}^*)$.

Proof. Existence of $\mathbf{x}^* \in \mathbb{S}^d$ such that $f(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$ follows from the proof of Proposition 3. Since $f(\mathbf{x}) \leq f(|\mathbf{x}|)$ and $S(|\mathbf{x}|) = S(\mathbf{x})$ for every $\mathbf{x} \in \mathfrak{R}^d$, $\tilde{\mathbf{x}}^* := |\mathbf{x}^*| \geq 0$ is also a maximizer of Q , i.e. $Q(\tilde{\mathbf{x}}^*) = \|f\|_{p_1, \dots, p_m}$. By definition, the singular vectors of f are the critical points of Q and thus $\tilde{\mathbf{x}}^*$ is a singular vector of f associated to the singular value $\|f\|_{p_1, \dots, p_m}$. \square

Theorem 13. Let $1 < p_1, \dots, p_m < \infty$ and $f \geq 0$ an irreducible tensor. If $\mathbf{x} \in \mathbb{S}_+^d$ is a singular vector of f , then $\mathbf{x} \in \mathbb{S}_{++}^d$. Moreover, there exists a singular vector $\mathbf{x}^* \in \mathbb{S}_{++}^d$ of f such that $Q(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$.

Proof. By Equation (1) we have $\sigma_i(\mathbf{x}) = \lambda^{p'_i-1} \mathbf{x}$ for every $i \in [m]$, where $\lambda = Q(\mathbf{x})$ is the singular value associated to \mathbf{x} . From Proposition 8 we know that, with $\alpha = \frac{1}{2}(1, \lambda^{1-p'_1}, \dots, \lambda^{1-p'_m})$, there is some $n \in \mathbb{N}$ such that $0 < T_\alpha^n(\mathbf{x}) = \mathbf{x}$. Existence of $\mathbf{x}^* > 0$ follows from Lemma 12 and the discussion above. \square

Using a similar argument as Friedland et al. (Theorem 3.3, [11]), we show that if there is $i \in [m]$ such that $(m-1)p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$ and $f \geq 0$ is weakly irreducible, then there exists a strictly positive singular vector associated to the maximal singular value of f . First, we recall a Theorem proved by Gaubert and Gunawardena.

Theorem 14 (Theorem 2, [16]). Let $F : \mathbb{R}_{++}^n \rightarrow \mathbb{R}_{++}^n$. For $t > 0$ and $J \subset [n]$ denote by $\mathbf{u}^J(t) = (u_1^J(t), \dots, u_n^J(t)) \in \mathbb{R}_{++}^n$ the following vector: $u_i^J(t) = 1 + t$ if $i \in J$ and $u_i^J(t) = 1$ if $i \notin J$. The di-graph $\mathcal{G}(F) = ([n], \mathcal{E}(F))$ associated to F is defined as follow: $(k, l) \in \mathcal{E}(F)$ if and only if $\lim_{t \rightarrow \infty} F_k(\mathbf{u}^{\{l\}}(t)) = \infty$. If $F(\mathbf{x}) \leq F(\mathbf{y})$ for every $0 < \mathbf{x} \leq \mathbf{y}$, F is positively 1-homogeneous and $\mathcal{G}(F)$ is strongly connected, then there exists $\lambda > 0$ and $\mathbf{x} \in \mathbb{R}_{++}^n$ such that $F(\mathbf{x}) = \lambda \mathbf{x}$.

Lemma 15. Let $1 < p_1, \dots, p_m < \infty$, $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ and $i, k \in [m]$ with $i \neq k$, then

$$s_{i,k}(\theta_1 \mathbf{x}_1, \dots, \theta_m \mathbf{x}_m) = \left(\prod_{l \in [m] \setminus \{i\}} \theta_l \right)^{p'_i(p'_k-1)} \theta_k^{1-p'_k} s_{i,k}(\mathbf{x}) \quad \forall \theta_1, \dots, \theta_m \in \mathbb{R}_{++}, \mathbf{x} \in \mathfrak{R}^{d-d_i}.$$

Proof. Follows from a straightforward computation. \square

Theorem 16. Let $1 < p_1, \dots, p_m < \infty$ and $f \geq 0$ be a weakly irreducible tensor. Suppose that there is $i \in [m]$ such that $(m-1)p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$, then f has a strictly positive singular vector $\mathbf{x}^* > 0$ so that $Q(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$.

Proof. Let $A_i : \mathfrak{R}_+^{d-d_i} \rightarrow \mathfrak{R}_+^{d-d_i}$ be defined by $A_i(\mathbf{x}) := (A_{i,1}(\mathbf{x}), \dots, A_{i,i-1}(\mathbf{x}), A_{i,i+1}(\mathbf{x}), \dots, A_{i,m}(\mathbf{x}))$ with

$$A_{i,k,j_k}(\mathbf{x}) := \left(x_{k,j_k}^{p_{\max} - p_k} \|\mathbf{x}_k\|_{p_k}^{p_k - p'_i(m-1)} \psi_{p_k}(s_{i,k,j_k}(\mathbf{x})) \right)^{\frac{1}{p_{\max}-1}} \quad \forall k \in [m] \setminus \{i\} \text{ and } j_k \in [d_k],$$

where $p_{\max} := \max_{k \in [m] \setminus \{i\}} p_k$. We say that a vector $\mathbf{x} \in \mathfrak{R}^{d-d_i}$ is an eigenvector of A_i if $\mathbf{x} \neq 0$ and there exists $\lambda \in \mathbb{R}$ such that $A_i(\mathbf{x}) = \lambda \mathbf{x}$. Suppose that we can prove the following.

- i) $A_i|_{\mathfrak{R}_{++}^{d-d_i}}$ satisfies all assumptions of Theorem 14.
- ii) If $\mathbf{x}^* \in \mathfrak{R}_{++}^{d-d_i}$ and $\mathbf{y} \in \mathfrak{R}_{++}^{d-d_i} \setminus \{0\}$ are such that $A_i(\mathbf{x}^*) = \lambda \mathbf{x}^*$ and $A_i(\mathbf{y}) = \mu \mathbf{y}$ then $\mu \leq \lambda$.
- iii) If $\mathbf{x} \in \mathfrak{R}_{++}^{d-d_i}$ is an eigenvector of A_i associated to a strictly positive eigenvalue λ , then it is a critical point of Q_i associated to the critical value $\lambda^{(p_{\max}-1)/p'_i}$.
- iv) If $\mathbf{x} \in \mathbb{S}_+^{d-d_i}$ is a critical point of Q_i associated to λ , then $A_i(\mathbf{x}) = \lambda^{p'_i/(p_{\max}-1)} \mathbf{x}$.

Using i), we may apply Theorem 14 and get the existence of $\mathbf{x}^* > 0$ such that $A_i(\mathbf{x}^*) = \lambda \mathbf{x}^*$. ii) ensures that the associated eigenvalue λ is maximal. By iii), we know that $\mathbf{x}^* > 0$ is a critical point of Q_i . Now, suppose by contradiction that $\lambda^{(p_{\max}-1)/p'_i} = Q_i(\mathbf{x}^*) < \|f\|_{p_1, \dots, p_m}$. Lemma 12 and Proposition 5 imply the existence of $\mathbf{y}^* \geq 0$ such that $Q_i(\mathbf{y}^*) = \|f\|_{p_1, \dots, p_m}$. By iv), we know that \mathbf{y}^* is an eigenvector of A_i associated to $\|f\|_{p_1, \dots, p_m}^{p'_i/(p_{\max}-1)} > \lambda$, a contradiction. Finally, since f is weakly irreducible we know from Corollary 11 that $(\|f\|_{p_1, \dots, p_m}, \tilde{\mathbf{x}}^*) := \Phi_i(\|f\|_{p_1, \dots, p_m}, \mathbf{x}^*)$ is such that $\tilde{\mathbf{x}}^* \in \mathfrak{R}^d$ is a strictly positive singular vector of f with $Q(\tilde{\mathbf{x}}^*) = \|f\|_{p_1, \dots, p_m}$. Now, we prove i) – iv).

- i) The fact that A_i is positively 1-homogeneous follows from Lemma 15. If $0 \leq \mathbf{x} \leq \mathbf{y}$, then $A_i(\mathbf{x}) \leq A_i(\mathbf{y})$ follows from $p_{\max} - p_k \geq 0, p_k - p'_i(m-1) \geq 0$ and Lemma 6. In order to prove the remaining property, that $\mathcal{G}(A_i)$ is strongly connected, we use the fact that f is weakly irreducible. So, let $\mathcal{G}(A_i) = (\mathcal{V}, \mathcal{E}(A_i))$ and $G(f) = (V, E(f))$ (see Definition 7 and Theorem 14) with $\mathcal{V} := (\{1\} \times [d_1]) \cup \dots \cup (\{i-1\} \times [d_{i-1}]) \cup (\{i+1\} \times [d_{i+1}]) \cup \dots \cup (\{m\} \times [d_m])$ and $V := (\{1\} \times [d_1]) \cup \dots \cup (\{m\} \times [d_m])$. We show that for every $k, l \in [m] \setminus \{i\}$, if $((k, j_k), (l, j_l)) \in E(f)$ then $((k, j_k), (l, j_l)) \in \mathcal{E}(A_i)$ and if $((k, j_k), (i, j_i)), ((i, j_i), (l, \nu_l)) \in E(f)$, then $((k, j_k), (l, \nu_l)) \in \mathcal{E}(A_i)$. Since $G(f)$ is an undirected connected graph (f is weakly irreducible) this would imply that $\mathcal{G}(A_i)$ is strongly connected. By definition, $((k, j_k), (l, j_l)) \in E(f)$ implies the existence of $j_s \in [d_s]$ for $s \in [m] \setminus \{k, l\}$ such that $f_{j_1, \dots, j_m} > 0$ and if $\mathbf{u}^{\{(l, j_l)\}}(t) > 0$ is defined as in Theorem 14, then $\partial_{k, j_k} f(\mathbf{u}^{\{(l, j_l)\}}(t)) \rightarrow \infty$ as $t \rightarrow \infty$. It follows by Proposition 8, b) and the definition of $s_{i, k}$ that $A_{i, k, j_k}(\mathbf{u}^{\{(l, j_l)\}}(t)) \rightarrow \infty$ as $t \rightarrow \infty$. Now, suppose that $((k, j_k), (i, j_i)), ((i, j_i), (l, \nu_l)) \in E(f)$, then there exists $j_s \in [d_s]$ and $\nu_t \in [d_t]$ for $s \in [m] \setminus \{k, i\}$ and $t \in [m] \setminus \{l, i\}$ such that $f_{j_1, \dots, j_m} > 0$ and $f_{\nu_1, \dots, \nu_{i-1}, j_i, \nu_{i+1}, \dots, \nu_m} > 0$. Again, let $\mathbf{u}^{\{(l, \nu_l)\}}(t) > 0$ be defined as in Theorem 14, then $f_{\nu_1, \dots, \nu_{i-1}, j_i, \nu_{i+1}, \dots, \nu_m} > 0$ implies that $\partial_{i, j_i} f(\mathbf{u}^{\{(l, \nu_l)\}}(t)) \rightarrow \infty$ as $t \rightarrow \infty$ and $f_{j_1, \dots, j_m} > 0$ implies that

$$\partial_{k, j_k} f\left(\mathbf{u}_1^{\{(l, \nu_l)\}}(t), \dots, \mathbf{u}_{i-1}^{\{(l, \nu_l)\}}(t), \psi_{p'_i}\left(\nabla_i f(\mathbf{u}^{\{(l, \nu_l)\}}(t))\right), \mathbf{u}_{i+1}^{\{(l, \nu_l)\}}(t), \dots, \mathbf{u}_m^{\{(l, \nu_l)\}}(t)\right) \rightarrow \infty,$$

as $t \rightarrow \infty$. It follows that $A_{i, k, j_k}(\mathbf{u}^{\{(l, \nu_l)\}}(t)) \rightarrow \infty$ as $t \rightarrow \infty$, i.e. $((k, j_k), (l, \nu_l)) \in \mathcal{E}(A_i)$. Thus A_i fulfills all assumptions of Theorem 14.

- ii) This result follows from Lemma 3.3 in [17]. We reproduce the proof here for convenience of the reader. Let $\mathbf{x}^* \in \mathfrak{R}_{++}^{d-d_i}$ and $\mathbf{y} \in \mathfrak{R}_{++}^{d-d_i} \setminus \{0\}$ be such that $A_i(\mathbf{x}^*) = \lambda \mathbf{x}^*$ and $A_i(\mathbf{y}) = \mu \mathbf{y}$. Set $\theta := \min \left\{ \frac{x_{k, j_k}^*}{y_{k, j_k}} \mid y_{k, j_k} > 0, k \in [m] \setminus \{i\}, j_k \in [d_k] \right\}$, then $\theta > 0$ and $\theta \mathbf{y} \leq \mathbf{x}^*$. We observed in i) that A_i is positively 1-homogeneous and $0 \leq \mathbf{x} \leq \mathbf{y}$ imply $A_i(\mathbf{x}) \leq A_i(\mathbf{y})$. It follows that for every $n \in \mathbb{N}$, we have $\theta \mu^n \mathbf{y} = \theta A_i^n(\mathbf{y}) = A_i^n(\theta \mathbf{y}) \leq A_i^n(\mathbf{x}^*) = \lambda^n \mathbf{x}$, where $A_i^k(\mathbf{z}) := A_i^{k-1}(A_i(\mathbf{z}))$ for $k \in \mathbb{N}, k > 1$ and $\mathbf{z} \geq 0$. This shows that if $\frac{\lambda}{\mu} < 1$, then $\theta \mathbf{y} \leq \lim_{n \rightarrow \infty} \left(\frac{\lambda}{\mu}\right)^n \mathbf{x} = 0$, a contradiction to $\mathbf{y} \in \mathfrak{R}_{++}^{d-d_i} \setminus \{0\}$.

- iii) Let $\mathbf{x} > 0$ be an eigenvector of A_i associated to the eigenvalue $\lambda > 0$. Note that $A_i(\mathbf{x}) = \lambda \mathbf{x}$ implies

$$\lambda^{p_{\max}-1} x_{k, j_k}^{p_k-1} = \|\mathbf{x}_k\|_{p_k}^{p_k-p'_i(m-1)} \psi_{p_k}(s_{i, k, j_k}(\mathbf{x})) \quad \forall k \in [m] \setminus \{i\}, j_k \in [d_k]. \quad (5)$$

Multiplying this equation by x_{k, j_k} and summing over $j_k \in [d_k]$ shows

$$\lambda^{p_{\max}-1} \|\mathbf{x}_k\|_{p_k}^{p_k} = \|\mathbf{x}_k\|_{p_k}^{p_k-p'_i(m-1)} \langle \psi_{p_k}(s_{i, k}(\mathbf{x})), \mathbf{x}_k \rangle = \|\mathbf{x}_k\|_{p_k}^{p_k-p'_i(m-1)} \|\nabla_i f(\mathbf{x})\|_{p'_i}^{p'_i} \quad \forall k \in [m] \setminus \{i\},$$

where we used $\langle \psi_{p_k}(s_{i,k}(\mathbf{x})), \mathbf{x}_k \rangle = f(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \sigma_i(\mathbf{x}), \mathbf{x}_{i+1}, \dots, \mathbf{x}_m) = \langle \psi_{p'_i}(\nabla_i f(\mathbf{x})), \nabla_i f(\mathbf{x}) \rangle$. Thus $\|\mathbf{x}_k\|_{p_k}^{p'_i(m-1)} = \|\nabla_i f(\mathbf{x})\|_{p'_i}^{p'_i} \lambda^{1-p_{\max}}$ for every $k \in [m] \setminus \{i\}$, i.e. $\|\mathbf{x}_1\|_{p_1} = \dots = \|\mathbf{x}_m\|_{p_m} =: \alpha$ and we have $\alpha^{-1}\mathbf{x} \in \mathbb{S}^{d-d_i}$. Now, substituting \mathbf{x} by $\tilde{\mathbf{x}} = \alpha^{-1}\mathbf{x}$ in Equation (5) and composing by $\psi_{p'_k}$ shows $\lambda^{(p_{\max}-1)(p'_k-1)} \tilde{\mathbf{x}}_{k,j_k} = s_{i,k,j_k}(\tilde{\mathbf{x}})$. By Equation (3) it follows that $\tilde{\mathbf{x}}$ is a critical point of Q_i . Since the critical points of Q_i are scale invariant, $\mathbf{x} = \alpha\tilde{\mathbf{x}}$ is also a critical point of Q_i and $Q_i(\mathbf{x}) = \lambda^{(p_{\max}-1)/p'_i}$.

- iv) Suppose that $\mathbf{x} \in \mathbb{S}_+^{d-d_i}$ is a critical point of Q_i and let $\lambda > 0$ its associated critical value. Since $\mathbf{x} \in \mathbb{S}^{d-d_i}$ is a singular vector of f , by Equation (3) we have $s_{i,k}(\mathbf{x}) = \lambda^{p'_i(p'_k-1)} \mathbf{x}_k$ for every $k \in [m] \setminus \{i\}$. It follows that $A_{i,k,j_k}(\mathbf{x}) = \left(x_{k,j_k}^{p_{\max}-p_k} \psi_{p_k}(\lambda^{p'_i(p'_k-1)} x_{k,j_k}) \right)^{1/(p_{\max}-1)} = \lambda^{p'_i/(p_{\max}-1)} x_{k,j_k}$, i.e. \mathbf{x} is an eigenvector of A_i associated to the eigenvalue $\lambda^{p'_i/(p_{\max}-1)} > 0$. \square

4 Collatz-Wielandt analysis and proof of the main Theorem

For $i \in [m]$, consider $\gamma_i^-, \gamma_i^+ : \mathbb{S}_{++}^{d-d_i} \rightarrow \mathbb{R}_{++}$, the Collatz-Wielandt ratios defined as follow:

$$\gamma_i^-(\mathbf{x}) := \prod_{k \in [m] \setminus \{i\}} \min_{j_k \in [d_k]} \left(\frac{s_{i,k,j_k}(\mathbf{x})}{x_{k,j_k}} \right)^{p_k-1} \quad \text{and} \quad \gamma_i^+(\mathbf{x}) := \prod_{k \in [m] \setminus \{i\}} \max_{j_k \in [d_k]} \left(\frac{s_{i,k,j_k}(\mathbf{x})}{x_{k,j_k}} \right)^{p_k-1}. \quad (6)$$

The next lemma is useful to prove the uniqueness of the strictly positive singular vector of f in our Perron-Frobenius Theorem.

Lemma 17. Let $f \geq 0$ be a weakly irreducible tensor, $i \in [m]$ and $\mathbf{x}, \mathbf{y} \in \mathfrak{R}^{d-d_i}$. Assume $0 < \mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$, furthermore consider the following sets for $\nu \in [m] \setminus \{i\}$: $J_\nu := \{j_\nu \in [d_\nu] \mid x_{\nu,j_\nu} = y_{\nu,j_\nu}\}$, $J_i := \{j_i \in [d_i] \mid \sigma_{i,j_i}(\mathbf{x}) = \sigma_{i,j_i}(\mathbf{y})\}$, $I_\nu := \{j_\nu \in [d_\nu] \mid s_{i,\nu,j_\nu}(\mathbf{x}) = s_{i,\nu,j_\nu}(\mathbf{y})\}$, $I_i := J_i$, $J := (\{1\} \times J_1) \cup \dots \cup (\{m\} \times J_m)$ and $I := (\{1\} \times I_1) \cup \dots \cup (\{m\} \times I_m)$. Then $I \subset J$ and if $J \neq \emptyset$ we have $I \neq J$.

Proof. First, we prove $I \subset J$. If $I = \emptyset$, there is nothing to prove, so suppose $I \neq \emptyset$. Let $(l, j_l) \in I$ with $l \in [m] \setminus \{i\}$, weak irreducibility of f implies the existence of $j_k \in [d_k]$ for each $k \in [m] \setminus \{l\}$ such that $f_{j_1, \dots, j_m} > 0$. So, $s_{i,l,j_l}(\mathbf{x}) = s_{i,l,j_l}(\mathbf{y})$ implies $\sigma_{i,j_l}(\mathbf{x}) = \sigma_{i,j_l}(\mathbf{y})$ because $0 < x_{k,j_k} \leq y_{k,j_k}$ for every $k \in [m] \setminus \{l, i\}$. Now, $\sigma_{i,j_l}(\mathbf{x}) = \sigma_{i,j_l}(\mathbf{y})$ implies $x_{l,j_l} = y_{l,j_l}$ since $0 < x_{k,j_k} \leq y_{k,j_k}$ for every $k \in [m] \setminus \{l, i\}$. Thus $(l, j_l) \in J$ and we have $I \subset J$. Now, suppose $J \neq \emptyset$, then $\mathbf{x} \neq \mathbf{y}$ implies the existence of $l \in [m] \setminus \{i\}$ and $j_l \in [d_l]$ such that $x_{l,j_l} < y_{l,j_l}$. Since f is weakly irreducible, there exists $j_k \in [d_k]$ for each $k \in [m] \setminus \{l\}$ such that $f_{j_1, \dots, j_m} > 0$. It follows that $\sigma_{i,j_l}(\mathbf{x}) < \sigma_{i,j_l}(\mathbf{y})$ and thus $I_i \neq [d_i]$ as well as $s_{i,k,j_k}(\mathbf{x}) < s_{i,k,j_k}(\mathbf{y})$ for $k \in [m] \setminus \{i\}$. Hence, $I_k \neq [d_k]$ for every $k \in [m]$. Now, on one hand, if there is $k \in [m]$ such that $J_k = [d_k]$, then $I_k \neq J_k$ and the proof is done. On the other hand, if there is $k \in [m]$ such that $J_k = \emptyset$, then $I_l = \emptyset$ for every $l \in [m] \setminus \{i\}$ and the proof is done. Finally, assume that $J_k \notin \{[d_k], \emptyset\}$ for every $k \in [m]$. Suppose by contradiction that $I = J$. Weak irreducibility of f implies the existence of a vertex between J and $V \setminus J$ in the graph $G(f) = (V, E(f))$, i.e. there exists $\nu, \mu \in [m]$, $\nu \neq \mu$, $j_\nu^* \in J_\nu$, $j_\mu^* \in [d_\mu] \setminus J_\mu$ and $j_k^* \in [d_k]$ for each $k \in [m] \setminus \{\nu, \mu\}$ such that $f_{j_1^*, \dots, j_m^*} > 0$. If $\nu \neq i$, $s_{i,\nu,j_\nu^*}(\mathbf{y}) = s_{i,\nu,j_\nu^*}(\mathbf{x})$ follows from $\nu \in J_\nu = I_\nu$ and thus $y_{\mu,j_\mu^*} = x_{\mu,j_\mu^*}$, a contradiction to $j_\mu^* \in [d_\mu] \setminus J_\mu$. If $\nu = i$, the equality $\sigma_{i,j_i^*}(\mathbf{y}) = \sigma_{i,j_i^*}(\mathbf{x})$ implies the same contradiction. \square

Note that the assumption $0 < \mathbf{x}$ in Lemma 17 can't be replaced by $0 \leq \mathbf{x}, S_i(\mathbf{x}) \neq 0$ as shown in the following example.

Example 18. Let $1 < p_1, p_2, p_3 < \infty$, $i = 3$ and $f \in \mathbb{R}^{2 \times 2 \times 2}$ the nonnegative weakly irreducible tensor defined by $f_{1,1,1} = f_{1,2,1} = f_{2,2,2} = 1$ and $f_{j_1, j_2, j_3} = 0$ else. Let $\mathbf{x} := ((1, 0), (1, 0))$ and $\mathbf{y} := ((1, 1), (1, 0)) \in \mathbb{R}^{(6-2)}$, then $0 \leq \mathbf{x} \leq \mathbf{y}$, $\mathbf{x} \neq \mathbf{y}$ and $S_i(\mathbf{x}) = \|(1, 0)\|_{p_1} \|(1, 0)\|_{p_2} = 1$. However, $s_{3,1}(\mathbf{x}) = (1, 0) = s_{3,1}(\mathbf{y})$ and $s_{3,2}(\mathbf{x}) = (1, 1) = s_{3,2}(\mathbf{y})$, thus, if I and J are defined as in Lemma 17, we get $J \subsetneq I$.

Theorem 19. Let f be a nonnegative weakly irreducible tensor and $1 < p_1, \dots, p_m < \infty$ are such that there exists $i \in [m]$ with $(m-1)p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$, therefore there exists $\mathbf{x}^* \in \mathbb{S}_{++}^{d-d_i}$ such that $Q_i(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$. Let γ_i^-, γ_i^+ as in Equation (6), then for every $\mathbf{z} \in \mathbb{S}_{++}^{d-d_i}$ we have $\gamma_i^-(\mathbf{z}) \leq \|f\|_{p_1, \dots, p_m}^{p'_i(m-1)} \leq \gamma_i^+(\mathbf{z})$ and equality holds if and only if $\mathbf{x}^* = \mathbf{z}$.

Proof. First of all, note that, by Theorem 16, we know that there exists some singular vector $\tilde{\mathbf{x}}^* \in \mathbb{S}^d$ of f such that $\tilde{\mathbf{x}}^* > 0$ and $f(\tilde{\mathbf{x}}^*) = \|f\|_{p_1, \dots, p_m}$. Using the bijection Φ_i of Proposition 5 and Corollary 11, we get the existence of $\mathbf{x}^* \in \mathbb{S}^{d-d_i}$ such that $\mathbf{x}^* > 0$ and $Q_i(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$. From Equation (3) and $\mathbf{x}^* \in \mathbb{S}_{++}^{d-d_i}$ follows $\left(\frac{s_{i,k,j_k}(\mathbf{x}^*)}{x_{k,j_k}^*} \right)^{p_k-1} = \|f\|_{p_1, \dots, p_m}^{p'_i}$ for all $k \in [m] \setminus \{i\}$ and every $j_k \in [d_k]$. In particular, this implies that $\gamma_i^-(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}^{p'_i(m-1)} = \gamma_i^+(\mathbf{x}^*)$. If $\mathbf{z} = \mathbf{x}^*$, then $\gamma_i^-(\mathbf{z}) = \gamma_i^-(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}^{p'_i(m-1)} = \gamma_i^+(\mathbf{x}^*) = \gamma_i^+(\mathbf{z})$ and the proof is done. Suppose that $\mathbf{z} \neq \mathbf{x}^*$ and for $l \in [m] \setminus \{i\}$, let $\theta_l := \min_{j_l \in [d_l]} \frac{z_{l,j_l}}{x_{l,j_l}^*}$ and $\Theta_l := \max_{j_l \in [d_l]} \frac{z_{l,j_l}}{x_{l,j_l}^*}$, then $\theta_l \mathbf{x}_l^* \leq \mathbf{z}_l \leq \Theta_l \mathbf{x}_l^*$ and $\theta_l = \|\theta_l \mathbf{x}_l^*\|_{p_l} \leq \|\mathbf{z}_l\|_{p_l} = 1 \leq \|\Theta_l \mathbf{x}_l^*\|_{p_l} = \Theta_l$, i.e $\theta_l \in [0, 1]$ and $\Theta_l \in [1, \infty]$. Observe that Lemmas 6 and 15 imply

$$\left(\prod_{l \in [m] \setminus \{i\}} \theta_l \right)^{p'_i} \theta_k^{-1} \psi_{p_k}(s_{i,k,j_k}(\mathbf{x}^*)) \leq \psi_{p_k}(s_{i,k,j_k}(\mathbf{z})) \leq \left(\prod_{l \in [m] \setminus \{i\}} \Theta_l \right)^{p'_i} \Theta_k^{-1} \psi_{p_k}(s_{i,k,j_k}(\mathbf{x}^*)).$$

Moreover,

$$\prod_{k \in [m] \setminus \{i\}} \left(\left(\prod_{l \in [m] \setminus \{i\}} \theta_l \right)^{p'_i} \theta_k^{-1} \right) = \left(\prod_{l \in [m] \setminus \{i\}} \theta_l \right)^{(m-1)p'_i} \left(\prod_{k \in [m] \setminus \{i\}} \theta_k^{-1} \right) = \prod_{l \in [m] \setminus \{i\}} \theta_l^{(m-1)p'_i-1}$$

and the same holds if we replace θ_l by Θ_l . The assumption $(m-1)p'_i \leq p_k$ guarantees $\Theta_k^{(m-1)p'_i-p_k} \leq 1 \leq \theta_k^{(m-1)p'_i-p_k}$ for every $k \in [m] \setminus \{i\}$. Now, if $\theta_l = 1$ for every $l \in [m] \setminus \{i\}$, then $\mathbf{z}_l = \mathbf{x}_l^*$ for every $l \in [m] \setminus \{i\}$, a contradiction to $\mathbf{x}^* \neq \mathbf{z}$. For the same reason, we can't have $\Theta_l = 1$ for every $l \in [m] \setminus \{i\}$. Hence, there is some $l, k \in [m] \setminus \{i\}$ such that $\theta_l < 1$ and $\Theta_k > 1$. Thus $\theta_l \mathbf{x}_l^* \neq \mathbf{z}_l$ and $\mathbf{z}_k \neq \Theta_k \mathbf{x}_k^*$. Applying Lemma 17 to $0 < (\theta_1 \mathbf{x}_1^*, \dots, \theta_m \mathbf{x}_m^*) \leq \mathbf{z}$ and $0 < \mathbf{z} \leq (\Theta_1 \mathbf{x}_1^*, \dots, \Theta_m \mathbf{x}_m^*)$, we get the existence of $\mu, \nu \in [m] \setminus \{i\}$, $j_\mu^+ \in [d_\mu]$ and $j_\nu^- \in [d_\nu]$ such that $\theta_\mu x_{\mu,j_\mu}^* = z_{\mu,j_\mu}^-$ and $\Theta_\nu x_{\nu,j_\nu}^* = z_{\nu,j_\nu}^+$, as well as

$$s_{i,\mu,j_\mu}^-(\theta_1 \mathbf{x}_1^*, \dots, \theta_m \mathbf{x}_m^*) < s_{i,\mu,j_\mu}^-(\mathbf{z}) \quad \text{and} \quad s_{i,\nu,j_\nu}^+(\mathbf{z}) < s_{i,\nu,j_\nu}^+(\Theta_1 \mathbf{x}_1^*, \dots, \Theta_m \mathbf{x}_m^*). \quad (7)$$

Furthermore, for each $l \in [m] \setminus \{\mu, \nu, i\}$ there exists indexes $j_l^+, j_l^- \in [d_l]$ such that $\theta_l x_{l,j_l}^* = z_{l,j_l}^-$ and $\Theta_l x_{l,j_l}^* = z_{l,j_l}^+$ by construction of θ_l and Θ_l . With Lemma 6, we get

$$\begin{aligned} \prod_{l \in [m] \setminus \{i\}} \frac{\psi_{p_l}(s_{i,l,j_l}^-(\mathbf{z}))}{\psi_{p_l}(z_{l,j_l}^-)} &> \left(\prod_{l \in [m] \setminus \{i\}} \theta_l^{(m-1)p'_i-p_l} \right) \left(\prod_{l \in [m] \setminus \{i\}} \frac{\psi_{p_l}(s_{i,l,j_l}^-(\mathbf{x}^*))}{\psi_{p_l}(x_{l,j_l}^*)} \right) \\ &\geq \left(\prod_{l \in [m] \setminus \{i\}} \frac{\psi_{p_l}(s_{i,l,j_l}^-(\mathbf{x}^*))}{\psi_{p_l}(x_{l,j_l}^*)} \right) = \|f\|_{p_1, \dots, p_m}^{p'_i(m-1)} > \prod_{l \in [m] \setminus \{i\}} \frac{\psi_{p_l}(s_{i,l,j_l}^+(\mathbf{z}))}{\psi_{p_l}(z_{l,j_l}^+)}, \end{aligned}$$

where we used Equation (7) for the strict inequalities. Observing that

$$\gamma_i^+(\mathbf{z}) \geq \prod_{l \in [m] \setminus \{i\}} \frac{\psi_{p_l}(s_{i,l,j_l}^-(\mathbf{z}))}{\psi_{p_l}(z_{l,j_l}^-)} = \prod_{l \in [m] \setminus \{i\}} \left(\frac{s_{i,l,j_l}^-(\mathbf{z})}{z_{l,j_l}^-} \right)^{p_l-1} \quad \text{and} \quad \prod_{l \in [m] \setminus \{i\}} \frac{\psi_{p_l}(s_{i,l,j_l}^+(\mathbf{z}))}{\psi_{p_l}(z_{l,j_l}^+)} \geq \gamma_i^-(\mathbf{z}),$$

shows $\gamma_i^+(\mathbf{z}) > \|f\|_{p_1, \dots, p_m}^{p'_i(m-1)} > \gamma_i^-(\mathbf{z})$. \square

Corollary 20. Let $f \geq 0$ be a weakly irreducible tensor and $1 < p_1, \dots, p_m < \infty$ are such that there exists $i \in [m]$ with $(m-1)p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$, therefore there exists $\mathbf{x}^* \in \mathbb{S}_{++}^{d-d_i}$ with $Q_i(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$. Then \mathbf{x}^* is the unique critical point of Q_i in $\mathbb{S}_{++}^{d-d_i}$. Moreover, if f is irreducible, then \mathbf{x}^* is the unique critical point of Q_i in $\mathbb{S}_+^{d-d_i}$.

Proof. Suppose that $\mathbf{y}^* \in \mathbb{S}_{++}^{d-d_i}$ is a critical point of Q_i . By Equation (3) we know that $\left(\frac{s_{i,k,j_k}(\mathbf{y}^*)}{y_{k,j_k}}\right)^{p_k-1} = Q_i(\mathbf{y}^*)^{p'_i}$ for every $k \in [m] \setminus \{i\}$ and $j_k \in [d_k]$. Thus, Theorem 19 implies $\mathbf{y}^* = \mathbf{x}^*$ since $Q_i(\mathbf{y}^*)^{p'_i(m-1)} = \gamma_i^-(\mathbf{y}^*) \leq \|f\|_{p_1, \dots, p_m}^{p'_i(m-1)} \leq \gamma_i^+(\mathbf{y}^*) = Q_i(\mathbf{y}^*)^{p'_i(m-1)}$. Now, suppose that f is irreducible. By Theorem 13 we know that every nonnegative singular vector of f is strictly positive. Since \mathbf{x}^* is the unique singular vector of f in \mathbb{S}_{++}^d , it is also the unique singular vector in \mathbb{S}_+^d . \square

As shown in the next example, there are weakly irreducible tensors with nonnegative singular vectors.

Example 21. Let $f \in \mathbb{R}^{2 \times 2 \times 2}$ be the tensor of Example 18 and $1 < p_1, p_2, p_3 < \infty$ such that there is $i \in [3]$ with $2p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$. Then, by Theorem 14, there exists a singular vector $\mathbf{x}^* \in \mathbb{S}_{++}^d$ of f . However, note that $\mathbf{y}^* := ((0, 1), (0, 1), (0, 1)) \in \mathbb{S}_+^d$ is also a singular vector of f .

Now, we discuss the assumptions made on p_1, \dots, p_m in Theorem 19.

Remark 22. Let $1 < p_1, \dots, p_m < \infty$ and $i \in [m]$, then

$$(m-1)p'_i \leq \min_{k \in [m] \setminus \{i\}} p_k \iff m-1 \leq (p_i-1) \left(\min_{k \in [m] \setminus \{i\}} p_k - (m-1) \right).$$

Furthermore, if there exists $i \in [m]$ with $(m-1)p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$, then at least one of the following condition is satisfied

- a) $(m-1)p'_i \leq p_k$, for every $k \in [m] \setminus \{i\}$, where $p_l = \min_{k \in [m]} p_k$,
- b) $(m-1)p'_n \leq p_k$, for every $k \in [m] \setminus \{i\}$, where $p_n = \max_{k \in [m]} p_k$.

Moreover, note that if $(m-1)p'_i \leq p_k$ for all $k \in [m] \setminus \{i\}$ and $m \geq 3$, then $2 \leq (m-1) \leq (m-1)p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$. In the case $m = 2$, we may also always choose $i \in [m]$ such that $p_k \geq 2$ for $k \in [2] \setminus \{i\}$ because $p'_1 \leq p_2$ is equivalent to $p'_2 \leq p_1$ which is true if and only if $1 \leq (p_1-1)(p_2-1)$. Thus we recover the matrix case as analyzed by Boyd [8]. Note however that our result implies that the strictly positive singular vector is unique. Bhaskara et al. [9] proved the uniqueness of the strictly positive singular vector but only for the case of strictly positive matrix.

Proof of Theorem 1. The min-max characterization follows from Theorem 19. The uniqueness par follows from Proposition 5 and Corollary 20. \square

5 Relation with other tensor spectral problems

It is well-known that if a matrix is symmetric, then its eigenvectors and singular vectors coincide up to sign. This observation can be extended for tensors as shown in the following proposition.

Proposition 23. Let $q_1, \dots, q_k \in \mathbb{N} \setminus \{0\}$ such that $q_1 + \dots + q_k = m$ and $f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ with $d_1 = \dots = d_{q_1} = \tilde{d}_1, d_{(q_1+1)} = \dots = d_{(q_1+q_2)} = \tilde{d}_2, \dots, d_{(q_1+\dots+q_{k-1}+1)} = \dots = d_{(q_1+\dots+q_k)} = \tilde{d}_k$. Suppose that f is partially symmetric in the sense that

$$f_{j_1, \dots, j_m} = f_{\sigma_1(j_1, \dots, j_{q_1}), \sigma_2(j_{(q_1+1)}, \dots, j_{(q_1+q_2)}), \dots, \sigma_k(j_{(q_1+\dots+q_{k-1}+1)}, \dots, j_{(q_1+\dots+q_k)})}$$

where, for $i = 1, \dots, k$, σ_i is any permutation of q_i elements. Then, every solution $(\lambda, \mathbf{x}_1, \dots, \mathbf{x}_k) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}^{\tilde{d}_1} \times \mathbb{R}^{\tilde{d}_2} \times \dots \times \mathbb{R}^{\tilde{d}_m}$ of the problem

$$\begin{cases} \nabla_{(q_1+\dots+q_{i-1}+1)} f(\mathbf{x}_1, \dots, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_2, \dots, \mathbf{x}_k, \dots, \mathbf{x}_k) = \lambda \psi_{\tilde{p}_i}(\mathbf{x}_i) \\ \|\mathbf{x}_1\|_{\tilde{p}_1} = \dots = \|\mathbf{x}_k\|_{\tilde{p}_k} = 1, \end{cases} \quad i = 1, \dots, k, \quad (8)$$

where $1 < \tilde{p}_1, \dots, \tilde{p}_k < \infty$, induces a ℓ^{p_1, \dots, p_m} singular vector of f with $p_1 = \dots = p_{q_1} = \tilde{p}_1, p_{(q_1+1)} = \dots = p_{(q_1+q_2)} = \tilde{p}_2, \dots, p_{(q_1+\dots+q_{k-1}+1)} = \dots = p_{(q_1+\dots+q_k)} = \tilde{p}_k$. In particular, if f and p_1, \dots, p_m satisfy the conditions of Theorem 1, then there is a unique strictly positive solution $(\lambda^*, \mathbf{x}_1^*, \dots, \mathbf{x}_k^*)$ to Problem (8). If f is irreducible, this solution is also the only nonnegative solution. Moreover, if $(\lambda^*, \tilde{\mathbf{x}}_1^*, \dots, \tilde{\mathbf{x}}_m^*)$ is the unique strictly positive ℓ^{p_1, \dots, p_m} -singular vector of f , then $(\lambda, \tilde{\mathbf{x}}_1^*, \tilde{\mathbf{x}}_{(q_1+1)}^*, \dots, \tilde{\mathbf{x}}_{(q_1+\dots+q_{k-1}+1)}^*)$ is the unique strictly positive solution of (8).

Proof. For $(\mathbf{y}_1, \dots, \mathbf{y}_k) \in \mathbb{R}^{\tilde{d}_1} \times \mathbb{R}^{\tilde{d}_2} \times \dots \times \mathbb{R}^{\tilde{d}_m}$, consider the injective map

$$\xi(\mathbf{y}_1, \dots, \mathbf{y}_k) := (\underbrace{\mathbf{y}_1, \dots, \mathbf{y}_1}_{q_1 \text{ times}}, \underbrace{\mathbf{y}_2, \dots, \mathbf{y}_2}_{q_2 \text{ times}}, \dots, \underbrace{\mathbf{y}_k, \dots, \mathbf{y}_k}_{q_k \text{ times}}) \in \mathbb{R}^{\tilde{d}_1} \times \dots \times \mathbb{R}^{\tilde{d}_m},$$

and the surjective map

$$\zeta(\mathbf{z}_1, \dots, \mathbf{z}_m) := (\mathbf{z}_1, \mathbf{z}_{(q_1+1)}, \dots, \mathbf{z}_{(q_1+\dots+q_{k-1}+1)}) \in \mathbb{R}^{\tilde{d}_1} \times \mathbb{R}^{\tilde{d}_2} \times \dots \times \mathbb{R}^{\tilde{d}_m}$$

defined for $(\mathbf{z}_1, \dots, \mathbf{z}_m) \in \mathbb{R}^{\tilde{d}_1} \times \dots \times \mathbb{R}^{\tilde{d}_m}$. Note that $\zeta(\xi(\mathbf{y})) = \mathbf{y}$ for every $\mathbf{y} \in \mathbb{R}^{\tilde{d}_1} \times \dots \times \mathbb{R}^{\tilde{d}_m}$. If $(\lambda, \mathbf{x}_1, \dots, \mathbf{x}_k)$ is a solution of (8), then the partial symmetry of f and the definition of ξ imply

$$\begin{aligned} \nabla_{(q_1+\dots+q_{i-1}+l)} f(\xi(\mathbf{x}_1, \dots, \mathbf{x}_k)) &= \nabla_{(q_1+\dots+q_{i-1}+1)} f(\xi(\mathbf{x}_1, \dots, \mathbf{x}_k)) = \lambda \psi_{\tilde{p}_i}(\mathbf{x}_i) \\ &= \lambda \psi_{p_i}(\xi(\mathbf{x}_1, \dots, \mathbf{x}_k)_{(q_1+\dots+q_{i-1}+l)}) \quad \forall l = 1, \dots, q_i, i = 1, \dots, k. \end{aligned}$$

This shows that $\xi(\mathbf{x}_1, \dots, \mathbf{x}_k)$ is a ℓ^{p_1, \dots, p_m} -singular vector of f associated to the singular value λ . On the other hand, if $\xi(\mathbf{x}_1, \dots, \mathbf{x}_k)$ is a ℓ^{p_1, \dots, p_m} -singular vector of f , then, from Equation (1), we know that $(\mathbf{x}_1, \dots, \mathbf{x}_k)$ is a solution to Problem (8). Now, suppose that f and p_1, \dots, p_m satisfy the conditions of Theorem 1. The existence of a strictly positive solution $(\lambda^*, \mathbf{x}_1^*, \dots, \mathbf{x}_k^*)$ to Problem (8), can be shown in the same way as Theorem 16 by considering the map $\tilde{A}_i: \mathbb{R}^{\tilde{d}_1} \times \dots \times \mathbb{R}^{\tilde{d}_k} \rightarrow \mathbb{R}^{\tilde{d}_1} \times \dots \times \mathbb{R}^{\tilde{d}_k}$ with $\tilde{A}_i := (\tilde{A}_{i,1}, \dots, \tilde{A}_{i,k})$ and $\tilde{A}_{i,k}(\mathbf{x}) := \zeta(A_{i,k}(\xi(\mathbf{x})))$ instead of the function A_i defined in the proof of Theorem 16 (note that the partial symmetry of f implies $\xi(\tilde{A}_{i,k}(\mathbf{x})) = A_{i,k}(\xi(\mathbf{x}))$ for every $\mathbf{x} \in \mathbb{R}^{\tilde{d}_1} \times \dots \times \mathbb{R}^{\tilde{d}_k}$). Now, we show the uniqueness of a strictly positive solution. Let $(\lambda, \mathbf{x}_1, \dots, \mathbf{x}_k), (\mu, \mathbf{u}_1, \dots, \mathbf{u}_k)$ be two strictly positive solutions to (8). By the uniqueness result of Theorem 1, we know that $\lambda = \mu$ and $\xi(\mathbf{x}_1, \dots, \mathbf{x}_k) = \xi(\mathbf{u}_1, \dots, \mathbf{u}_k)$. $(\lambda, \mathbf{x}_1, \dots, \mathbf{x}_k) = (\mu, \mathbf{u}_1, \dots, \mathbf{u}_k)$, follows then from the injectivity of ξ and thus there can be only one positive solution to Problem (8). If f is irreducible, a similar argument shows that there can be only one nonnegative solution. Finally, suppose that $(\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_m)$ is the unique strictly positive ℓ^{p_1, \dots, p_m} -singular vector of f and assume that there exists a strictly positive solution $(\mathbf{x}_1^*, \dots, \mathbf{x}_k^*)$ to (8), we must have $\xi(\mathbf{x}_1^*, \dots, \mathbf{x}_k^*) = (\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_m)$ and thus $\zeta(\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_m) = (\mathbf{x}_1^*, \dots, \mathbf{x}_k^*)$. \square

Note that the partial symmetry assumption of Proposition 23 is crucial as shown by the following example.

Example 24. Let $f \in \mathbb{R}^{2 \times 2 \times 2}$ with $f_{1,2,2} = f_{2,1,2} = 0$ and $f_{i,k,l} = 1$ else. Let $1 < p < \infty, k = 2, q_1 = 1$ and $q_2 = 2$. Set $\mathbf{x}_1 = \mathbf{x}_2 := (\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}})$, then $\nabla_1 f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2) = \nabla_2 f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2) = \frac{3}{2^{3/p-1}} \psi_p(\frac{1}{2^{1/p}}, \frac{1}{2^{1/p}})$ and thus $(\frac{3}{2^{3/p-1}}, \mathbf{x}_1, \mathbf{x}_2)$ is a solution of (8). However $\nabla_3 f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2) = \frac{1}{2^{3/p-1}} \psi_p(2^{\frac{p+1}{p(p-1)}}, 2^{\frac{1}{p(p-1)}})$, i.e. $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2)$ is not a $\ell^{p,p,p}$ -singular value of f .

Problem (8) models many of the eigenvalue problems for tensors. In particular if $k = 1$, then we recover the H -eigenvalue problem for $p = m$ and the Z -eigenvalue problem for $p = 2$. These problems

Higher-order Generalized Power Method (HGPM)	
Input:	$f \in \mathbb{R}^{d_1 \times \dots \times d_m}$ and p_1, \dots, p_m satisfying assumptions of Theorem 1, $\varepsilon > 0$.
Initialization:	$i \in [m]$ with $(m-1)p'_i \leq p_k$ for all $k \in [m] \setminus \{i\}$. (if $m = 2$, choose $i \in [m]$ with $p_i \leq p_k$ for $k \in [2] \setminus \{i\}$), $\mathbf{x}^0 > 0$ with $\ \mathbf{x}_l^0\ _{p_l} = 1$ for $l \in [m]$, $k = 0$.
Do	
	$\mathbf{z}^k = (s_{i,1}(\mathbf{x}^k), \dots, s_{i,i-1}(\mathbf{x}^k), s_{i,i+1}(\mathbf{x}^k), \dots, s_{i,m}(\mathbf{x}^k))$
	$\lambda_-^{k+1} = \prod_{l \in [m] \setminus \{i\}} \min_{j_l \in [d_l]} \left(\frac{z_{l,j_l}^k}{x_{l,j_l}^k} \right)^{\frac{p_l-1}{p'_i(m-1)}}, \quad \lambda_+^{k+1} = \prod_{l \in [m] \setminus \{i\}} \max_{j_l \in [d_l]} \left(\frac{z_{l,j_l}^k}{x_{l,j_l}^k} \right)^{\frac{p_l-1}{p'_i(m-1)}}$
	$\mathbf{x}^{k+1} = \left(\frac{\mathbf{z}_1^k}{\ \mathbf{z}_1^k\ _{p_1}}, \dots, \frac{\mathbf{z}_{i-1}^k}{\ \mathbf{z}_{i-1}^k\ _{p_{i-1}}}, \frac{\mathbf{z}_{i+1}^k}{\ \mathbf{z}_{i+1}^k\ _{p_{i+1}}}, \dots, \frac{\mathbf{z}_m^k}{\ \mathbf{z}_m^k\ _{p_m}} \right)$
	$k = k + 1$
Until	$(\lambda_+^k - \lambda_-^k) < \varepsilon$
Output:	Approximate of the maximal singular vector $\mathbf{x} := (\mathbf{x}_1^k, \dots, \mathbf{x}_{i-1}^k, \frac{\sigma_i(\mathbf{x}^k)}{\ \sigma_i(\mathbf{x}^k)\ _{p_i}}, \mathbf{x}_{i+1}^k, \dots, \mathbf{x}_m^k)$ and approximate of the maximal singular value $\lambda := f(\mathbf{x})$. Moreover, $\left \frac{\lambda_-^k + \lambda_+^k}{2} - \ f\ _{p_1, \dots, p_m} \right < \varepsilon$.

σ_i is defined in Eq. (2), p. 4 and $s_{i,k}$ is defined in Eq. (4), p. 5, to select $i \in [m]$ see Remark 22.

were introduced in 2005 by Qi [18]. Still for $k = 1$, the more general problem for $1 < p < \infty$ is known as ℓ^p -eigenvalue problem and was introduced by Lim in [4]. In [10], [4] and [11], a Perron-Frobenius Theorem is proved for ℓ^p -eigenvalues of tensors. The requirement on p is $p \geq m$ and is equivalent to the condition of Theorem 1 when $p_1 = \dots = p_m$. The case $k = 2$ and $\tilde{p}_1 = \tilde{p}_2 = 2$, is known as M -eigenvalue problem and was introduced by Chang, Qi and Zhou in [19]. The more general formulation for $k = 2$ and $1 < \tilde{p}_1, \tilde{p}_2 < \infty$ is known as $\ell^{\tilde{p}_1, \tilde{p}_2}$ -singular value problem for rectangular tensors and was introduced by Ling and Qi in [12]. Ling and Qi also proved a Perron-Frobenius Theorem for $\ell^{\tilde{p}_1, \tilde{p}_2}$ -singular value problems and the condition on \tilde{p}_1, \tilde{p}_2 is $\tilde{p}_1, \tilde{p}_2 \geq m$. This condition is equivalent to ours whenever $q_1 \notin \{1, m-1\}$, nevertheless, for the case $q_1 = 1$, we only require $m-1 \leq (\tilde{p}_1-1)(\tilde{p}_2-m-1)$ and for the case $q_1 = m-1$, our condition becomes $m-1 \leq (\tilde{p}_2-1)(\tilde{p}_1-m-1)$.

6 Computation of the tensor norm and singular vectors of a nonnegative tensor

We derive now an algorithm which takes benefit of the theory developed above. More precisely, motivated by the properties of γ_i^-, γ_i^+ in Equation (6), we study the sequence $(\mathbf{x}^k)_{k \in \mathbb{N}} \subset \mathbb{S}^{d-d_i}$ produced by the Higher-order Generalized Power Method (HGPM). The HGPM is a tensor generalization of the algorithm proposed by Boyd in [8] for matrices (they coincide for $m = 2$). In order to prove the convergence of the sequence $(\mathbf{x}^k)_{k \in \mathbb{N}}$ produced by HGPM, we show first that if the starting vector \mathbf{x}^0 is close enough to the singular vector $\mathbf{x}^* > 0$ then we have linear convergence. Then, we show that for any starting point $\mathbf{x}^0 \in \mathbb{S}_{++}^{d-d_i}$ the sequence converges to \mathbf{x}^* . Let $G : \mathbb{S}_{++}^{d-d_i} \rightarrow \mathbb{S}_{++}^{d-d_i}$ be defined by

$$G(\mathbf{x}) := \left(\frac{s_{i,1}(\mathbf{x})}{\|s_{i,1}(\mathbf{x})\|_{p_1}}, \dots, \frac{s_{i,m}(\mathbf{x})}{\|s_{i,m}(\mathbf{x})\|_{p_m}} \right).$$

Note that if f is weakly irreducible, by Corollary 10, G is well defined and for $k \in \mathbb{N}$ we have $\mathbf{x}^{k+1} = G(\mathbf{x}^k)$ where $(\mathbf{x}^k)_{k \in \mathbb{N}}$ is the sequence produced by HGPM. Furthermore, if \mathbf{x}^* is a strictly positive critical point of Q_i , then $G(\mathbf{x}^*) = \mathbf{x}^*$. The next proposition gives some properties of the sequences $(\lambda_-^k)_{k \in \mathbb{N}}, (\lambda_+^k)_{k \in \mathbb{N}}$ produced by HGPM that motivate our choice for the stopping criterium.

Proposition 25. Let $f \in \mathbb{R}^{d_1 \times \dots \times d_m}, f \geq 0$ be a weakly irreducible tensor, $1 < p_1, \dots, p_m < \infty$ such that there exists $i \in [m]$ with $(m-1)p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$ and γ_i^-, γ_i^+ as in Equation (6). Furthermore, consider the sequences $(\lambda_-^k)_{k \in \mathbb{N}}$ and $(\lambda_+^k)_{k \in \mathbb{N}}$ produced by HGPM, then

$$\lambda_-^k \leq \lambda_-^{k+1} \leq \|f\|_{p_1, \dots, p_m} \leq \lambda_+^{k+1} \leq \lambda_+^k \quad \forall k \in \mathbb{N},$$

and when the Algorithm stops we have $\left| \frac{\lambda_-^k + \lambda_+^k}{2} - \|f\|_{p_1, \dots, p_m} \right| < \varepsilon$.

Proof. The proof is in two steps, first we prove that $\gamma_i^-(\mathbf{z}) \leq \gamma_i^-(G(\mathbf{z})) \leq \|f\|_{p_1, \dots, p_m}^{p'_i(m-1)} \leq \gamma_i^+(G(\mathbf{z})) \leq \gamma_i^+(\mathbf{z})$ for every $\mathbf{z} \in \mathbb{S}_{++}^{d-d_i}$ and then we conclude the proof. Let $k \in [m] \setminus \{i\}, \phi_k := \min_{j_k \in [d_k]} \frac{s_{i,k,j_k}(\mathbf{z})}{z_{k,j_k}}$ and $\varphi_k := \max_{j_k \in [d_k]} \frac{s_{i,k,j_k}(\mathbf{z})}{z_{k,j_k}}$, then we have $\gamma_i^-(\mathbf{z}) = \prod_{k \in [m] \setminus \{i\}} \phi_k^{p_k-1}, \mathbf{z}_k \phi_k \leq s_{i,k}(\mathbf{z}), \gamma_i^+(\mathbf{z}) = \prod_{k \in [m] \setminus \{i\}} \varphi_k^{p_k-1}$ and $\mathbf{z}_k \varphi_k \geq s_{i,k}(\mathbf{z})$. Note that

$$0 < \frac{\phi_k}{\|s_{i,k}(\mathbf{z})\|_{p_k}} = \frac{\|\mathbf{z}_k \phi_k\|_{p_k}}{\|s_{i,k}(\mathbf{z})\|_{p_k}} \leq \frac{\|s_{i,k}(\mathbf{z})\|_{p_k}}{\|s_{i,k}(\mathbf{z})\|_{p_k}} = 1 \leq \frac{\|\mathbf{z}_k \varphi_k\|_{p_k}}{\|s_{i,k}(\mathbf{z})\|_{p_k}} = \frac{\varphi_k}{\|s_{i,k}(\mathbf{z})\|_{p_k}} < \infty,$$

follows from $\mathbf{z} \in \mathbb{S}^{d-d_i}$. We only prove the inequality for γ_i^- since the other inequality can be proved in the same way. By Lemmas 6 and 15, for any $l \in [m] \setminus \{i\}$ and any $j_l \in [d_l]$ we have

$$\psi_{p_l}(s_{i,l,j_l}(G(\mathbf{z}))) \geq \left(\frac{\phi_l}{\|s_{i,l}(\mathbf{z})\|_{p_l}} \right)^{-1} \left(\prod_{k \in [m] \setminus \{i\}} \frac{\phi_k}{\|s_{i,k}(\mathbf{z})\|_{p_k}} \right)^{p'_i} \psi_{p_l}(s_{i,l,j_l}(\mathbf{z})).$$

With $\frac{\phi_k}{\|s_{i,k}(\mathbf{z})\|_{p_k}} \leq 1$ and $1 < p'_i(m-1) \leq p_k$ for every $k \in [m] \setminus \{i\}$, we get

$$\prod_{k \in [m] \setminus \{i\}} \left(\frac{\phi_k}{\|s_{i,k}(\mathbf{z})\|_{p_k}} \right)^{p'_i(m-1)-1} \geq \prod_{k \in [m] \setminus \{i\}} \left(\frac{\phi_k}{\|s_{i,k}(\mathbf{z})\|_{p_k}} \right)^{p_k-1} = \gamma_i^-(\mathbf{z}) \prod_{k \in [m] \setminus \{i\}} \frac{1}{\|s_{i,k}(\mathbf{z})\|_{p_k}^{p_k-1}},$$

Combining these facts shows that for every $j_1 \in [d_1], \dots, j_m \in [d_m]$ holds

$$\gamma_i^-(\mathbf{z}) \leq \prod_{l \in [m] \setminus \{i\}} \frac{\psi_{p_l}(s_{i,l,j_l}(G(\mathbf{z})))}{\frac{\psi_{p_l}(s_{i,l,j_l}(\mathbf{z}))}{\|s_{i,l}(\mathbf{z})\|_{p_l}^{p_l-1}}} = \prod_{l \in [m] \setminus \{i\}} \frac{(s_{i,l,j_l}(\mathbf{z}^s))^{p_l-1}}{\left(\frac{s_{i,l,j_l}(\mathbf{z})}{\|s_{i,l}(\mathbf{z})\|_{p_l}} \right)^{p_l-1}} = \prod_{l \in [m] \setminus \{i\}} \left(\frac{s_{i,l,j_l}(G(\mathbf{z}))}{(G(\mathbf{z}))_{l,j_l}} \right)^{p_l-1}.$$

Take the minimum over $j_1 \in [d_1], \dots, j_m \in [d_m]$ to get $\gamma_i^-(\mathbf{z}) \leq \gamma_i^-(G(\mathbf{z}))$. By Theorem 19, we know that $\gamma_i^-(G(\mathbf{z})) \leq \|f\|_{p_1, \dots, p_m}^{p'_i(m-1)} \leq \gamma_i^+(G(\mathbf{z}))$. This concludes the first step of our proof. Now, if $(\mathbf{x}^k)_{k \in \mathbb{N}}$ is the sequence produced by HGPM, then, by Corollary 10, $(\mathbf{x}^k)_{k \in \mathbb{N}} \subset \mathbb{S}_{++}^{d-d_i}$ since $\mathbf{x}^0 \in \mathbb{S}_{++}^{d-d_i}$ by assumption. Moreover, note that $\mathbf{x}^{k+1} = G(\mathbf{x}^k), \lambda_-^{k+1} = (\gamma_i^-(\mathbf{x}^k))^{\frac{1}{p'_i(m-1)}}$ and $\lambda_+^{k+1} = (\gamma_i^+(\mathbf{x}^k))^{\frac{1}{p'_i(m-1)}}$ for every $k \in \mathbb{N}$. It follows that $\lambda_-^k \leq \lambda_-^{k+1} \leq \|f\|_{p_1, \dots, p_m} \leq \lambda_+^{k+1} \leq \lambda_+^k$ for every $k \in \mathbb{N}$. Finally, if $(\lambda_+^k - \lambda_-^k) < \varepsilon$, subtracting $\frac{\lambda_+^k - \lambda_-^k}{2}$ from the inequality $\lambda_-^k \leq \|f\|_{p_1, \dots, p_m} \leq \lambda_+^k$, shows

$$-\varepsilon < -\frac{\lambda_+^k - \lambda_-^k}{2} \leq \|f\|_{p_1, \dots, p_m} \leq \frac{\lambda_+^k - \lambda_-^k}{2} < \varepsilon. \quad \square$$

Now, we prove the convergence of the sequences produced by HGPM.

Lemma 26. Let $f \geq 0$ be a weakly irreducible tensor and $1 < p_1, \dots, p_m < \infty$ such that there exists $i \in [m]$ with $(m-1)p'_i \leq p_k$ for all $k \in [m] \setminus \{i\}$. Let $\mathbf{x}^* \in \mathbb{S}_{++}^{d-d_i}$ be a critical point of Q_i and $\lambda := Q_i(\mathbf{x}^*)$. Furthermore, consider the function $F : \mathfrak{R}^{d-d_i} \rightarrow \mathfrak{R}^{d-d_i}$ defined by $F(\mathbf{x}) := (F_1(\mathbf{x}), \dots, F_m(\mathbf{x}))$ where for each $k \in [m] \setminus \{i\}$, $F_k(\mathbf{x}) := \lambda^{1-p'_i(p'_k-1)} \|\mathbf{x}_k\|_{p_k}^{p'_k+(m-1)p'_i(1-p'_k)+\rho} s_{i,k}(\mathbf{x})$ and $\rho > 0$ is such that $p'_i + (m-1)p'_i(1-p'_i) + \rho > 0$ for all $l \in [m] \setminus \{i\}$. Let us denote by B the Jacobian matrix of F at \mathbf{x}^* . Then the function F is positively $(\rho+1)$ -homogeneous, B is primitive with Perron-root $\lambda_1 := (\rho+1)\lambda$ and for any $\mathbf{g}, \mathbf{h} \in \mathfrak{R}^{d-d_i}, k \in [m] \setminus \{i\}$, we have

$$\langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{h}_k, (B\mathbf{g})_k \rangle = \langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{g}_k, (B\mathbf{h})_k \rangle,$$

where \circ denotes the Hadamard product and $|\mathbf{y}|^q := (|y_1|^q, \dots, |y_n|^q)$ for $\mathbf{y} \in \mathbb{R}^n, q \in \mathbb{R}$.

Proof. Since $\mathbf{x}^* > 0$, there exists some open neighborhood $U \subset \mathfrak{R}_{++}^{d-d_i}$ of \mathbf{x}^* such that $s_{i,k}$ is smooth on U for any $k \in [m] \setminus \{i\}$. Existence of such a neighborhood is obvious since $f(\mathbf{x})$ and $\nabla_i f(\mathbf{x})$ are smooth on \mathfrak{R}^d . Furthermore, f is weakly irreducible thus, by Proposition 8, we have $\partial_{k,j_k} f(\mathbf{x}) > 0$ for any $\mathbf{x} > 0$ and $\psi_p(t)$ is smooth on $\mathbb{R} \setminus \{0\}$. With help of Lemma 15 it is straightforward to check that F is positively $(\rho+1)$ -homogeneous. Let $\alpha_k(\mathbf{x}) := \lambda^{1-p'_i(p'_k-1)} \|\mathbf{x}_k\|_{p_k}^{p'_k+(m-1)p'_i(1-p'_k)+\rho}$, then for every $\mathbf{x} \in U$ and $l \in [m] \setminus \{i\}$, we have $\nabla_l \alpha_k(\mathbf{x}) = \lambda^{1-p'_i(p'_k-1)} (p'_k + (m-1)p'_i(1-p'_k) + \rho) \|\mathbf{x}_k\|_{p_k}^{p'_k+(m-1)p'_i(1-p_k)+\rho-p_k} \psi_{p_k}(\mathbf{x}_k)$ if $l = k$ and $\nabla_l \alpha_k(\mathbf{x}) = 0$ if $k \neq l$. In particular, $\mathbf{x}^* \in \mathbb{S}_{++}^d$ thus $\nabla_k \alpha_k(\mathbf{x}^*) > 0$. Note that for every $k, l \in [m] \setminus \{i\}, j_l \in [d_l]$ and $j_k \in [d_k]$, we have $\partial_{l,j_l} F_{k,j_k}(\mathbf{x}) = s_{i,k,j_k}(\mathbf{x}) \partial_{l,j_l} \alpha_k(\mathbf{x}) + \alpha_k(\mathbf{x}) \partial_{l,j_l} s_{i,k,j_k}(\mathbf{x})$. Moreover, using $(p' - 2) = (2 - p)(p' - 1)$ for $1 < p < \infty$, a tedious computation shows

$$\begin{aligned} \partial_{l,j_l} s_{i,k,j_k}(\mathbf{x}^*) &= (p'_i - 1)(p'_k - 1) \lambda^{p'_i(p'_k-1)-2} |x_{k,j_k}^*|^{2-p_k} \left\langle |\mathbf{x}_i^*|^{2-p_i} \circ \nabla_i \partial_{l,j_l} f(\mathbf{x}^*), \nabla_i \partial_{k,j_k} f(\mathbf{x}^*) \right\rangle \\ &\quad + (p'_k - 1) \lambda^{p'_i(p'_k-1)-1} |x_{k,j_k}^*|^{2-p_k} \partial_{l,j_l} \partial_{k,j_k} f(\mathbf{x}^*) \quad \forall k, l \in [m], j_k \in [d_k], j_l \in [d_l]. \end{aligned}$$

In particular, note that $\partial_{l,j_l} s_{i,k,j_k}(\mathbf{x}^*) \geq 0$ follows from $f \geq 0$ and $\mathbf{x}^* > 0$, thus $B \geq 0$ because $s_{i,k,j_k}(\mathbf{x}^*) \geq 0$ by Lemma 6, $\alpha_k(\mathbf{x}^*) \geq 0$ and $\partial_{l,j_l} \alpha_k(\mathbf{x}^*) \geq 0$. From Equation (3) and $\mathbf{x}^* \in \mathbb{S}^d$, we know that $s_{i,k}(\mathbf{x}^*) = \lambda^{p'_i(p'_k-1)} \mathbf{x}_k^* > 0$ and $\partial_{k,j_k} \alpha_k(\mathbf{x}^*) = (p'_k + (m-1)p'_i(1-p'_k) + \rho) \lambda^{1-p'_i(p'_k-1)} \psi_{p_k}(x_{k,j_k}^*) > 0$, thus

$$\begin{aligned} B_{(k,j_k),(k,l_k)} &= s_{i,k,j_k}(\mathbf{x}^*) \partial_{k,l_k} \alpha_k(\mathbf{x}^*) + \alpha_k(\mathbf{x}^*) \partial_{k,l_k} s_{i,k,j_k}(\mathbf{x}^*) \\ &\geq s_{i,k,j_k}(\mathbf{x}^*) \partial_{k,l_k} \alpha_k(\mathbf{x}^*) = \lambda (p'_k + (m-1)p'_i(1-p'_k) + \rho) (x_{k,j_k}^*)^{p_k} > 0 \end{aligned}$$

for all $k \in [m] \setminus \{i\}$ and $j_k, l_k \in [d_k]$. This shows that the matrix B has strictly positive blocks of size $d_k \times d_k$ on its main diagonal. In order to prove that B is an irreducible matrix, we show that for every $k, l \in [m] \setminus \{i\}$ with $k \neq l$ there exists $j_k \in [d_k]$ and $j_l \in [d_l]$ such that $\max \{B_{(k,j_k),(l,j_l)}, B_{(l,j_l),(k,j_k)}\} > 0$. This would imply that the graph associated to the adjacency matrix B is connected. Fix $k, l \in [m] \setminus \{i\}$ with $k \neq l$ and suppose by contradiction that for every $j_k \in [d_k]$ and every $j_l \in [d_l]$, we have $0 = B_{(k,j_k),(l,j_l)} = B_{(l,j_l),(k,j_k)}$. It follows that $\partial_{l,j_l} s_{i,k,j_k}(\mathbf{x}^*) = 0$ and $\partial_{l,j_l} \partial_{k,j_k} f(\mathbf{x}^*) = 0$ for every $j_k \in [d_k], j_l \in [d_l]$. So, $0 = \partial_{l,j_l} f(\mathbf{x}^*) = \langle \mathbf{x}_k^*, \nabla_k \partial_{l,j_l} f(\mathbf{x}^*) \rangle$ for every $j_l \in [d_l]$, thus $0 = \langle \mathbf{x}_l, \nabla_l f(\mathbf{x}^*) \rangle = f(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$, a contradiction. Hence the graph associated to the nonnegative adjacency matrix B is connected and B has strictly positive main diagonal entries, it follows from Lemma 8.5.5 and Theorem 8.5.2 in [20] that B is primitive. Now, using $s_{i,k}(\mathbf{x}^*) = \lambda^{p'_i(p'_k-1)} \mathbf{x}_k^*$ it can be observed that $F(\mathbf{x}^*) = \lambda \mathbf{x}^*$. Approximating $\alpha \mapsto F(\alpha \mathbf{x}^*)$ linearly at 1 shows $\alpha^{1+\rho} \lambda \mathbf{x}^* = \alpha^{1+\rho} F(\mathbf{x}^*) = F(\alpha \mathbf{x}^*) = F(\mathbf{x}^*) + (\alpha - 1)B\mathbf{x}^* + o(\alpha - 1)$ for every $\alpha > 1$. Subtract $\lambda \mathbf{x}^* + (\alpha - 1)B\mathbf{x}^*$ on both sides and take the limit $\alpha \rightarrow 1$ to get

$$(1 + \rho)\lambda \mathbf{x}^* - B\mathbf{x}^* = \lim_{\alpha \downarrow 1} \frac{(\alpha^{1+\rho} - 1)\lambda \mathbf{x}^* - (\alpha - 1)B\mathbf{x}^*}{(\alpha - 1)} = 0,$$

i.e. $(1 + \rho)\lambda > 0$ is the Perron-root of B associated to the strictly positive eigenvector \mathbf{x}^* . Now, we prove that $\langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{h}_k, (B\mathbf{g})_k \rangle = \langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{g}_k, (B\mathbf{h})_k \rangle$. Since F is differentiable on U , for every $\mathbf{h} \in \mathfrak{R}^{d-d_i}$ we

have $(B\mathbf{h})_k = (\delta F(\mathbf{x}^*; \mathbf{h}))_k = \delta F_k(\mathbf{x}^*; \mathbf{h})$ where $\delta F(\mathbf{x}; \mathbf{h}) := \lim_{\varepsilon \rightarrow 0} \varepsilon^{-1} (F(\mathbf{x} + \varepsilon \mathbf{h}) - F(\mathbf{x}))$ is the directional derivative of F at \mathbf{x} in the direction \mathbf{h} . The multiplication rule for directional derivative, $\mathbf{x}^* \in \mathbb{S}^{d-d_i}$ and $s_{i,k}(\mathbf{x}^*) = \lambda^{p'_i(p'_k-1)} \mathbf{x}^*$, show

$$\delta F_k(\mathbf{x}^*; \mathbf{h}) = \alpha_k(\mathbf{x}^*) \delta s_{i,k}(\mathbf{x}^*; \mathbf{h}) + s_{i,k}(\mathbf{x}^*) \delta \alpha_k(\mathbf{x}^*; \mathbf{h}) = \lambda^{1-p'_i(p'_k-1)} \delta s_{i,k}(\mathbf{x}^*; \mathbf{h}) + \lambda^{p'_i(p'_k-1)} \mathbf{x}_k^* \langle \nabla \alpha_k(\mathbf{x}^*), \mathbf{h} \rangle.$$

Let $C := \lambda^{1-p'_i(p'_k-1)} (p'_k + (m-1)p'_i(1-p_k) + \rho)$, then

$$\langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{g}_k, \mathbf{x}_k^* \langle \nabla \alpha_k(\mathbf{x}^*), \mathbf{h} \rangle \rangle = C \langle \psi_{p_k}(\mathbf{x}_k^*), \mathbf{h}_k \rangle \langle \psi_{p_k}(\mathbf{x}_k^*), \mathbf{g}_k \rangle = \langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{h}_k, \mathbf{x}_k^* \langle \nabla \alpha_k(\mathbf{x}^*), \mathbf{g} \rangle \rangle.$$

In order to conclude the proof, we show $\langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{h}_k, \delta s_{i,k}(\mathbf{x}^*; \mathbf{g}) \rangle = \langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{g}_k, \delta s_{i,k}(\mathbf{x}^*; \mathbf{h}) \rangle$. Another tedious computation shows $\delta s_{i,k}(\mathbf{x}; \mathbf{h}) = (p'_k - 1) |s_{i,k}(\mathbf{x})|^{2-p_k} \circ L_{i,k}(\mathbf{x}; \mathbf{h})$ for all $\mathbf{x} \in U$ where

$$\begin{aligned} L_{i,k}(\mathbf{x}, \mathbf{h}) := & \sum_{l \in [m] \setminus \{k, i\}} \nabla_k f \left(\mathbf{x}_1, \dots, \mathbf{x}_{l-1}, \mathbf{h}_l, \mathbf{x}_{l+1}, \dots, \mathbf{x}_{i-1}, \psi_{p'_i}(\nabla_i f(\mathbf{x})), \mathbf{x}_{i+1}, \dots, \mathbf{x}_m \right) \\ & + (p'_i - 1) \sum_{s \in [m] \setminus \{k\}} \nabla_k f \left(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, |\nabla_i f(\mathbf{x})|^{p'_i-2} \circ \nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{s-1}, \mathbf{h}_s, \mathbf{x}_{s+1}, \dots, \mathbf{x}_m), \mathbf{x}_{i+1}, \dots, \mathbf{x}_m \right). \end{aligned}$$

Note that for any $k, l \in [m] \setminus \{i\}$ with $k \neq l$ holds

$$\begin{aligned} \langle \nabla_k f(\mathbf{x}_1, \dots, \mathbf{x}_{l-1}, \mathbf{h}_l, \mathbf{x}_{l+1}, \dots, \mathbf{x}_m), \mathbf{g}_k \rangle &= f(\mathbf{x}_1, \dots, \mathbf{x}_{l-1}, \mathbf{h}_l, \mathbf{x}_{l+1}, \dots, \mathbf{x}_{k-1}, \mathbf{g}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_m) \\ &= \langle \nabla_l f(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{g}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_m), \mathbf{h}_l \rangle, \end{aligned}$$

furthermore for $s \in [m] \setminus \{i\}$, we have

$$\begin{aligned} & \left\langle \nabla_k f \left(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, |\nabla_i f(\mathbf{x})|^{p'_i-2} \circ \nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{s-1}, \mathbf{h}_s, \mathbf{x}_{s+1}, \dots, \mathbf{x}_m), \mathbf{x}_{i+1}, \dots, \mathbf{x}_m \right), \mathbf{g}_k \right\rangle \\ &= f \left(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, |\nabla_i f(\mathbf{x})|^{p'_i-2} \circ \nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{s-1}, \mathbf{h}_s, \mathbf{x}_{s+1}, \dots, \mathbf{x}_m), \mathbf{x}_{i+1}, \dots, \mathbf{x}_{k-1}, \mathbf{g}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_m \right) \\ &= \left\langle \nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{g}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_m), |\nabla_i f(\mathbf{x})|^{p'_i-2} \circ \nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{s-1}, \mathbf{h}_s, \mathbf{x}_{s+1}, \dots, \mathbf{x}_m) \right\rangle \\ &= \left\langle |\nabla_i f(\mathbf{x})|^{p'_i-2} \circ \nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{g}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_m), \nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{s-1}, \mathbf{h}_s, \mathbf{x}_{s+1}, \dots, \mathbf{x}_m) \right\rangle \\ &= \left\langle \nabla_s f \left(\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, |\nabla_i f(\mathbf{x})|^{p'_i-2} \circ \nabla_i f(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{g}_k, \mathbf{x}_{k+1}, \dots, \mathbf{x}_m), \mathbf{x}_{i+1}, \dots, \mathbf{x}_m \right), \mathbf{h}_s \right\rangle. \end{aligned}$$

These relations imply $\langle L_{i,k}(\mathbf{x}; \mathbf{h}), \mathbf{g}_k \rangle = \langle L_{i,k}(\mathbf{x}; \mathbf{g}), \mathbf{h}_k \rangle$ for all $\mathbf{x} \in U$. Thus, for every $k \in [m] \setminus \{i\}$, we have

$$\langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{g}_k, \delta s_{i,k}(\mathbf{x}^*; \mathbf{h}) \rangle = (p'_k - 1) \lambda^{p'_i(p'_k-2)} \langle L_{i,k}(\mathbf{x}; \mathbf{h}), \mathbf{g}_k \rangle = \langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{h}_k, \delta s_{i,k}(\mathbf{x}^*; \mathbf{g}) \rangle. \quad \square$$

Proposition 27. Let $f \geq 0$ a weakly irreducible tensor, $1 < p_1, \dots, p_m < \infty$ such that there is $i \in [m]$ with $(m-1)p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$, \mathbf{x}^* the unique strictly positive critical point of Q_i in \mathbb{S}^{d-d_i} . Moreover, let $DG(\mathbf{x}^*)$ be the Jacobian matrix of G at \mathbf{x}^* , then the spectral radius of $DG(\mathbf{x}^*)$ is strictly smaller than 1, i.e. $\rho(DG(\mathbf{x}^*)) < 1$.

Proof. Let B as in Lemma 26, then B is primitive and has Perron root $\lambda_1 = (\rho + 1) \|f\|_{p_1, \dots, p_m}$. Let $d := d_1 + \dots + d_m$ and $\lambda_1, \dots, \lambda_{d-d_i}$ be the eigenvalues of B . We may order them so that $\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_{d-d_i}|$. We claim that $|\lambda_2|$ is the spectral radius $\rho(M)$ of the matrix $M := \lambda_1 DG(\mathbf{x}^*)$. Let $U \subset \mathfrak{R}_{++}^{d-d_i}$ an open neighborhood of \mathbf{x}^* such that F and G are smooth in U . Observe that Corollary 10 implies $F_k(\mathbf{x}) > 0$ for every $\mathbf{x} \in U$ and each $k \in [m] \setminus \{i\}$. Moreover, $G(\mathbf{x}) = \left(\frac{F_1(\mathbf{x})}{\|F_1(\mathbf{x})\|_{p_1}}, \dots, \frac{F_m(\mathbf{x})}{\|F_m(\mathbf{x})\|_{p_m}} \right)$ for every $\mathbf{x} \in U$ and thus, in particular, $G(\mathbf{x}^*) = \mathbf{x}^*$. A straightforward computation shows that $(M\mathbf{h})_l = (B\mathbf{h})_l - \mathbf{x}_l^* \langle \psi_{p_l}(\mathbf{x}_l^*), (B\mathbf{h})_l \rangle$ for every $l \in [m] \setminus \{i\}$ and every $\mathbf{h} \in \mathfrak{R}^{d-d_i}$. It follows that $(M\mathbf{x}^*)_l = (B\mathbf{x}^*)_l - \mathbf{x}_l^* \langle \psi_{p_l}(\mathbf{x}_l^*), (B\mathbf{x}^*)_l \rangle = \lambda_1 \mathbf{x}_l^* - \lambda_1 \mathbf{x}_l^* \langle \psi_{p_l}(\mathbf{x}_l^*), \mathbf{x}_l^* \rangle = 0$, i.e. \mathbf{x}^* is an eigenvector of M associated to the eigenvalue 0. First, suppose that the eigenvalues of B are all distinct and different from 0. Now, from Proposition 26, we know that for

every \mathbf{h}, \mathbf{g} holds $\langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{h}_k, (B\mathbf{g})_k \rangle = \langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{g}_k, (B\mathbf{h})_k \rangle$. For $s \in [d], s \geq 2$ let us denote by \mathbf{u}^s the eigenvector of B such that $B\mathbf{u}^s = \lambda_s \mathbf{u}^s$, then

$$\lambda_s \langle \psi_{p_k}(\mathbf{x}_k^*), \mathbf{u}_k^s \rangle = \langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{x}_k^*, (B\mathbf{u}^s)_k \rangle = \langle |\mathbf{x}_k^*|^{p_k-2} \circ \mathbf{u}_k^s, (B\mathbf{x}^*)_k \rangle = \lambda_1 \langle \psi_{p_k}(\mathbf{x}_k^*), \mathbf{u}_k^s \rangle.$$

From $\lambda_s \neq 0$ and $\lambda_1 \neq \lambda_s$ follows $\langle \psi_{p_k}(\mathbf{x}_k^*), \mathbf{u}_k^s \rangle = 0$. This shows that for every $k \in [m] \setminus \{i\}$ we have $(M\mathbf{u}^s)_k = \lambda_s \mathbf{u}_k^s - \mathbf{x}^* \lambda_s \langle \psi_{p_k}(\mathbf{x}_k^*), \mathbf{u}_k^s \rangle = \lambda_s \mathbf{u}_k^s$, i.e. λ_s is an eigenvalue of M associated to the eigenvector \mathbf{u}^s . Thus, the eigenvalues of M are exactly $0, \lambda_2, \dots, \lambda_{d-d_i}$ and we have $\rho(M) = |\lambda_2|$. It follows that $\rho(DG(\mathbf{x}^*)) = \frac{|\lambda_2|}{\lambda_1} < 1$. The case when B does not satisfy the above assumption is treated as in Corollary 5.2, [11]. \square

Corollary 28. Let $f, p_1, \dots, p_m, \mathbf{x}^*$ and G as in Proposition 27. Then, there exists a norm $\|\cdot\|_G$ on \mathfrak{R}^{d-d_i} , $r_0 > 0$ and $0 < \nu < 1$ such that for every $\mathbf{y}^0 \in \mathfrak{R}^{d-d_i}$ with $\|\mathbf{y}^0 - \mathbf{x}^*\|_G < r_0$ we have

$$\|\mathbf{y}^{k+1} - \mathbf{x}^*\|_G \leq \nu \|\mathbf{y}^k - \mathbf{x}^*\|_G \quad \forall k \in \mathbb{N}, \quad (9)$$

where $\mathbf{y}^{k+1} := G(\mathbf{y}^k)$ for every $k \in \mathbb{N}$.

Proof. Let U be an open neighborhood of \mathbf{x}^* such that G is differentiable on U and $\mathbf{u} > 0$ for every $\mathbf{u} \in U$. By Proposition 27 we know that $\rho(DG(\mathbf{x}^*)) < 1$, so let $L, \nu > 0$ such that $\rho(DG(\mathbf{x}^*)) < L < \nu < 1$. It is a classical result (see Lemma 5.6.10, [20]) that there exists a norm $\|\cdot\|_G$ on \mathfrak{R}^{d-d_i} such that for every $\mathbf{v} \in \mathfrak{R}^{d-d_i}$ holds $\|DG(\mathbf{x}^*)\mathbf{v}\|_G \leq L\|\mathbf{v}\|_G$. Since G is differentiable at \mathbf{x}^* , we have $G(\mathbf{u}) = G(\mathbf{x}^*) + DG(\mathbf{x}^*)(\mathbf{u} - \mathbf{x}^*) + R(\mathbf{u}, \mathbf{x}^*)$, with $\lim_{\mathbf{u} \rightarrow \mathbf{x}^*} \frac{\|R(\mathbf{u}, \mathbf{x}^*)\|_G}{\|\mathbf{u} - \mathbf{x}^*\|_G} = 0$. Let $\varepsilon > 0$ be such that $\varepsilon \leq \nu - L$, then there is $r_0 > 0$ such that for every $\mathbf{u} \in \mathbb{S}_{++}^{d-d_i}$ with $\|\mathbf{u} - \mathbf{x}^*\|_G < r_0$ we have $\|R(\mathbf{u}, \mathbf{x}^*)\|_G \leq \varepsilon \|\mathbf{u} - \mathbf{x}^*\|_G$. By decreasing r_0 if necessary, we may suppose that $\|\mathbf{u} - \mathbf{x}^*\|_G < r_0$ implies $\mathbf{u} \in U$. It follows that for every $\mathbf{y}^0 \in \mathbb{S}^{d-d_i}$ with $\|\mathbf{y}^0 - \mathbf{x}^*\|_G < r_0$, it holds

$$\begin{aligned} \|\mathbf{y}^1 - \mathbf{x}^*\|_G &= \|G(\mathbf{y}^0) - G(\mathbf{x}^*)\|_G = \|DG(\mathbf{x}^*)(\mathbf{y}^0 - \mathbf{x}^*) + R(\mathbf{y}^0, \mathbf{x}^*)\|_G \\ &\leq \|DG(\mathbf{x}^*)(\mathbf{y}^0 - \mathbf{x}^*)\|_G + \|R(\mathbf{y}^0, \mathbf{x}^*)\|_G \leq (L + \varepsilon)\|\mathbf{y}^0 - \mathbf{x}^*\|_G \leq \nu \|\mathbf{y}^0 - \mathbf{x}^*\|_G. \end{aligned}$$

Since $\nu < 1$, we have $\|\mathbf{y}^1 - \mathbf{x}^*\|_G \leq \nu \|\mathbf{y}^0 - \mathbf{x}^*\|_G < \|\mathbf{y}^0 - \mathbf{x}^*\|_G < r_0$. Thus we may apply recursively the above argument to get $\|\mathbf{y}^{k+1} - \mathbf{x}^*\|_G \leq \nu \|\mathbf{y}^k - \mathbf{x}^*\|_G$ for every $k \in \mathbb{N}$. \square

It follows directly that if \mathbf{x}^0 is close enough to the strictly positive singular vector \mathbf{x}^* of f (i.e. $\|\mathbf{x}^0 - \mathbf{x}^*\|_G < r_0$ with the notations of Corollary 28), the sequence $(\mathbf{x}^k)_{k \in \mathbb{N}}$ produced by HGPM has \mathbf{x}^* as its limit and the convergence rate is linear. However, this result is of little practical use if we don't know \mathbf{x}^* . This is why we show now that for every $\mathbf{x}^0 \in \mathbb{S}_{++}^{d-d_i}$, the sequence $(\mathbf{x}^k)_{k \in \mathbb{N}}$ converges to \mathbf{x}^* . In order to do it, we use a Lemma proved by R. Nussbaum [21].

Lemma 29 (Lemma 2.3, [21]). Let (S, μ_S) be a metric space with metric μ_S and suppose that S_0 is a connected subset of S and μ is a metric on S_0 which gives the same topology on S_0 as that inherited from S . Let $T : S_0 \rightarrow S_0$ be a map such that $\mu(T(\mathbf{x}), T(\mathbf{y})) \leq \mu(\mathbf{x}, \mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in S_0$. For $n \in \mathbb{N}$ and $\mathbf{x} \in S_0$, let $T^n(\mathbf{x}) := T^{n-1}(T(\mathbf{x}))$. Assume that there exists $\mathbf{x}^* \in S_0$ and an open neighborhood U of \mathbf{x}^* such that $U \cap S_0 \neq \emptyset$ and $\lim_{n \rightarrow \infty} \mu_S(T^n(\mathbf{x}), \mathbf{x}^*) = 0$ for every $\mathbf{x} \in U \cap S_0$. Finally, if there exists a continuous map $\varphi : \{t \in \mathbb{R} \mid t \geq 0\} \rightarrow \{t \in \mathbb{R} \mid t \geq 0\}$ with $\varphi(0) = 0$ such that $\mu_S(\mathbf{x}, \mathbf{y}) \leq \varphi(\mu(\mathbf{x}, \mathbf{y}))$ for all $\mathbf{x}, \mathbf{y} \in U \cap S_0$. Then $\lim_{n \rightarrow \infty} \mu_S(T^n(\mathbf{x}), \mathbf{x}^*) = 0$ for every $\mathbf{x} \in S_0$.

We will apply Lemma 29 with $S_0 = \mathbb{S}_{++}^{d-d_i}$, $S = \mathfrak{R}^{d-d_i}$, $\mu_S(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_G$, $T = G$ and $U = \{\mathbf{x} \in \mathfrak{R}^{d-d_i} \mid \|\mathbf{x} - \mathbf{x}^*\|_G < r_0\}$ where $\|\cdot\|_G$ and $r_0 > 0$ are such that Equation (9) is satisfied. Now, we build the metric μ on S_0 such that $\mu(G(\mathbf{x}), G(\mathbf{y})) \leq \mu(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{S}_{++}^{d-d_i}$ and prove that the topology on $(\mathbb{S}_{++}^{d-d_i}, \mu)$ is the same as that inherited from $(\mathfrak{R}^{d-d_i}, \mu_S)$.

Proposition 30. Let $1 < p_1, \dots, p_m < \infty, i \in [m]$ and $\mu: \mathbb{S}_{++}^{d-d_i} \times \mathbb{S}_{++}^{d-d_i} \rightarrow \mathbb{R}$ defined by

$$\mu(\mathbf{x}, \mathbf{y}) := \ln \left(\prod_{l \in [m] \setminus \{i\}} \frac{\max_{j_l \in [d_l]} \left(\frac{x_{l,j_l}}{y_{l,j_l}} \right)^{p_l-1}}{\min_{j_l \in [d_l]} \left(\frac{x_{l,j_l}}{y_{l,j_l}} \right)^{p_l-1}} \right),$$

then $(\mathbb{S}_{++}^{d-d_i}, \mu)$ is a metric space.

Proof. For $l \in [m]$, let $\mathbb{S}_{++}^{d_l} := \{\mathbf{x}_l \in \mathbb{R}_{++}^{d_l} \mid \|\mathbf{x}_l\|_{p_l} = 1\}$ and $\mu_l, \mu_l^-, \mu_l^+: \mathbb{S}_{++}^{d_l} \times \mathbb{S}_{++}^{d_l} \rightarrow \mathbb{R}$ with

$$\mu_l^-(\mathbf{x}_l, \mathbf{y}_l) := \min_{j_l \in [d_l]} \left(\frac{x_{l,j_l}}{y_{l,j_l}} \right)^{p_l-1}, \mu_l^+(\mathbf{x}_l, \mathbf{y}_l) := \max_{j_l \in [d_l]} \left(\frac{x_{l,j_l}}{y_{l,j_l}} \right)^{p_l-1} \text{ and } \mu_l(\mathbf{x}_l, \mathbf{y}_l) := \ln \left(\frac{\mu_l^+(\mathbf{x}_l, \mathbf{y}_l)}{\mu_l^-(\mathbf{x}_l, \mathbf{y}_l)} \right). \quad (10)$$

It follows from Theorem 1.2 in [21] that $(\mathbb{S}_{++}^{d_l}, \mu_l)$ is a complete metric space. Note that $\mathbb{S}_{++}^{d-d_i} = \mathbb{S}_{++}^{d_1} \times \dots \times \mathbb{S}_{++}^{d_{i-1}} \times \mathbb{S}_{++}^{d_{i+1}} \times \dots \times \mathbb{S}_{++}^{d_m}$ and $\mu(\mathbf{x}, \mathbf{y}) = \sum_{l \in [m] \setminus \{i\}} \mu_l(\mathbf{x}_l, \mathbf{y}_l)$, i.e. $(\mathbb{S}_{++}^{d-d_i}, \mu)$ is the product metric space of $(\mathbb{S}_{++}^{d_1}, \mu_1), \dots, (\mathbb{S}_{++}^{d_{i-1}}, \mu_{i-1}), (\mathbb{S}_{++}^{d_{i+1}}, \mu_{i+1}), \dots, (\mathbb{S}_{++}^{d_m}, \mu_m)$ and thus is a metric space itself. \square

Proposition 31. Let $f \geq 0$ be a weakly irreducible and $1 < p_1, \dots, p_m < \infty$ are such that there is $i \in [m]$ with $(m-1)p'_i \leq p_k$ for all $k \in [m] \setminus \{i\}$ and, if $m = 2$, choose i such that $p_i \leq p_k$ for $k \in [2] \setminus \{i\}$. Let $\|\cdot\|_G$ be a norm such that Equation (9) is satisfied and μ the metric of Proposition 30. Then, for every $\mathbf{u} \in \mathbb{S}_{++}^{d-d_i}$, there exist $r > 0$ and $c, C > 0$ such that

$$c\|\mathbf{u} - \mathbf{x}\|_G \leq \mu(\mathbf{u}, \mathbf{x}) \leq C\|\mathbf{u} - \mathbf{x}\|_G \quad \forall \mathbf{x} \in \{\mathbf{x} \in \mathfrak{R}^{d-d_i} \mid \|\mathbf{x} - \mathbf{u}\|_G < r\} \cap \mathbb{S}_{++}^{d-d_i}.$$

Note that the assumption $p_i \leq p_k$ for $k \in [m] \setminus \{i\}$ when $m = 2$ is not restrictive by Remark 22.

Proof. Let $l \in [m] \setminus \{i\}$ and $\mathbf{v}_l \in \mathbb{S}_{++}^{d_l}$, note that since $\mathbb{R}_+ \rightarrow \mathbb{R}_+: t \mapsto t^{p_l-1}$ is an increasing function ($p_l > 1$ by assumption), for every $\mathbf{x}_l, \mathbf{y}_l \in \mathbb{R}_{++}^{d_l}$, we have

$$\mu_l(\mathbf{x}_l, \mathbf{y}_l) = (p_l - 1) \ln \left(\frac{\max_{j_l \in [d_l]} \frac{x_{l,j_l}}{y_{l,j_l}}}{\min_{j_l \in [d_l]} \frac{x_{l,j_l}}{y_{l,j_l}}} \right).$$

There exists $\xi_l > 0$ such that $B_{\xi_l}^l(\mathbf{v}_l) \subset \mathbb{R}_{++}^{d_l}$. Equation (1.21) in [21] reads

$$\frac{1}{p_l - 1} \mu(\mathbf{v}_l, \mathbf{x}_l) =: \tilde{\mu}_l(\mathbf{v}_l, \mathbf{x}_l) \leq \ln \left(\frac{\xi_l + \|\mathbf{v}_l - \mathbf{x}_l\|_{p_l}}{\xi_l - \|\mathbf{v}_l - \mathbf{x}_l\|_{p_l}} \right) = \ln \left(\frac{1 + \frac{\|\mathbf{v}_l - \mathbf{x}_l\|_{p_l}}{\xi_l}}{1 - \frac{\|\mathbf{v}_l - \mathbf{x}_l\|_{p_l}}{\xi_l}} \right) \quad \forall \mathbf{x}_l \in B_{\xi_l}^l(\mathbf{v}_l).$$

So, let $t(\mathbf{x}_l) := \frac{\|\mathbf{v}_l - \mathbf{x}_l\|_{p_l}}{\xi_l}$ and $0 < \varepsilon_l < \xi_l$, then $\overline{B_{\varepsilon_l}^l(\mathbf{v}_l)} \subset B_{\xi_l}^l(\mathbf{v}_l)$ and we have $\tilde{\mu}_l(\mathbf{v}_l, \mathbf{x}_l) \leq \ln \left(\frac{1+t(\mathbf{x}_l)}{1-t(\mathbf{x}_l)} \right)$ for all $\mathbf{x}_l \in \overline{B_{\varepsilon_l}^l(\mathbf{v}_l)}$. Now, the function $h: [0, \frac{\varepsilon_l}{\xi_l}] \rightarrow \mathbb{R}, t \mapsto \ln \left(\frac{1+t}{1-t} \right)$ is continuously differentiable on $[0, \frac{\varepsilon_l}{\xi_l}]$ and $2 \leq h'(t) = 2(1-t^2)^{-1} \leq 2 \left(1 - (\varepsilon_l/\xi_l)^2 \right)^{-1}$ for all $t \in [0, \frac{\varepsilon_l}{\xi_l}]$. It follows that h is a Lipschitz function and thus there exists $K_1 > 0$ such that $|h(t) - h(s)| \leq K_1|t - s|$ for every $s, t \in [0, \frac{\varepsilon_l}{\xi_l}]$. Thus

$$\tilde{\mu}_l(\mathbf{v}_l, \mathbf{x}_l) \leq h(t(\mathbf{x}_l)) = |h(t(\mathbf{x}_l)) - h(0)| \leq K_1|t(\mathbf{x}_l) - t(\mathbf{v}_l)| = \frac{K_1}{\xi_l} \|\mathbf{x}_l - \mathbf{v}_l\|_{p_l} \quad \forall \mathbf{x}_l \in \overline{B_{\varepsilon_l}^l(\mathbf{v}_l)}.$$

In Equation (1.20) in [21] it is shown

$$\|\mathbf{x}_l - \mathbf{y}_l\|_{p_l} \leq 3 \left(e^{\tilde{\mu}_l(\mathbf{x}_l, \mathbf{y}_l)} - 1 \right), \quad \forall \mathbf{x}_l, \mathbf{y}_l \in \mathbb{S}_{++}^{d_l}. \quad (11)$$

In particular, as shown above, for every $\mathbf{x}_l \in \overline{B_{\varepsilon_l}^l(\mathbf{v}_l)}$ we have $\tilde{\mu}(\mathbf{v}_l, \mathbf{x}_l) \leq \frac{\varepsilon_l K_1}{\xi_l}$ and the function $t \mapsto e^t$ is Lipschitz on $[0, \frac{\varepsilon_l K_1}{\xi_l}]$ (as it is smooth and the derivative is bounded on the interval). So, there exists $K_2 > 0$ such that $|e^s - e^t| \leq \frac{K_2}{3}|s - t|$ for every $s, t \in [0, \frac{\varepsilon_l K_1}{\xi_l}]$. It follows that

$$\|\mathbf{x}_l - \mathbf{v}_l\|_{p_l} \leq 3|e^{\tilde{\mu}_l(\mathbf{v}_l, \mathbf{x}_l)} - 3e^0| \leq K_2|\tilde{\mu}_l(\mathbf{v}_l, \mathbf{x}_l) - 0| = K_2\tilde{\mu}_l(\mathbf{v}_l, \mathbf{x}_l) \quad \forall \mathbf{x}_l \in \mathbb{S}_{++}^{d_l} \cap \overline{B_{\varepsilon_l}^l(\mathbf{v}_l)}.$$

Now, with $\tilde{K} := \max\{K_2, \frac{(p_l-1)K_1}{\xi_l}\}$, since $p_l \geq 2$ for $l \in [m] \setminus \{i\}$ by Remark 22, we get

$$\|\mathbf{x}_l - \mathbf{v}_l\|_{p_l} \leq \tilde{K}(p_l - 1)\tilde{\mu}_l(\mathbf{v}_l, \mathbf{x}_l) = \tilde{K}\mu_l(\mathbf{v}_l, \mathbf{x}_l) \leq \tilde{K}^2\|\mathbf{x}_l - \mathbf{v}_l\|_{p_l} \quad \forall \mathbf{x}_l \in \mathbb{S}_{++}^{d_l} \cap \overline{B_{\varepsilon_l}^l(\mathbf{v}_l)}.$$

So, for every $l \in [m] \setminus \{i\}$ there exists $r_l > 0$ and $C_l > 0$ such that $\|\mathbf{u}_l - \mathbf{x}_l\|_{p_l} \leq C_l\mu_l(\mathbf{x}_l, \mathbf{u}_l) \leq C_l^2\|\mathbf{u}_l - \mathbf{x}_l\|_{p_l}$ for all $\mathbf{x}_l \in B_{r_l}^l(\mathbf{u}_l) \cap \mathbb{S}_{++}^{d_l}$. Now, note that $\|\mathbf{v}\|_{\bar{p}} := \sum_{l \in [m] \setminus \{i\}} \|\mathbf{v}_l\|_{p_l}$ is a norm on \mathfrak{R}^{d-d_i} . Since all norms are equivalent on finite dimensional spaces, there exists a constant C_0 such that $\|\mathbf{v}\|_{\bar{p}} \leq C_0\|\mathbf{v}\|_G \leq C_0^2\|\mathbf{v}\|_{\bar{p}}$ for every $\mathbf{v} \in \mathfrak{R}^{d-d_i}$. Let $r = \min_{l \in [m] \setminus \{i\}} \frac{r_l}{C_0}$ and $C = C_0 \max_{l \in [m] \setminus \{i\}} C_l$, then for all $\mathbf{x} \in \{\mathbf{z} \in \mathfrak{R}^{d-d_i} \mid \|\mathbf{z} - \mathbf{u}\|_G < r\} \cap \mathbb{S}_{++}^{d-d_i}$ we have $\|\mathbf{x}_l - \mathbf{u}_l\|_{p_l} \leq \|\mathbf{x} - \mathbf{u}\|_{\bar{p}} \leq C_0\|\mathbf{x} - \mathbf{u}\|_G < r_l$ for every $l \in [m] \setminus \{i\}$, and thus

$$\begin{aligned} \|\mathbf{u} - \mathbf{x}\|_G &\leq C_0\|\mathbf{u} - \mathbf{x}\|_{\bar{p}} \leq C_0 \sum_{l \in [m] \setminus \{i\}} C_l\mu_l(\mathbf{u}_l, \mathbf{x}_l) \leq C\mu(\mathbf{u}, \mathbf{x}) = C \sum_{l \in [m] \setminus \{i\}} \mu_l(\mathbf{u}_l, \mathbf{x}_l) \\ &\leq C \sum_{l \in [m] \setminus \{i\}} C_l\|\mathbf{u}_l - \mathbf{x}_l\|_{p_l} \leq C\|\mathbf{u} - \mathbf{x}\|_{\bar{p}} \max_{l \in [m] \setminus \{i\}} C_l \leq C^2\|\mathbf{u} - \mathbf{x}\|_G. \end{aligned}$$

Divide the inequality by C to conclude the proof. \square

Now, we prove that G is non-expansive with respect to the metric μ defined in Proposition 30.

Proposition 32. Let $f \geq 0$ a weakly irreducible tensor, $1 < p_1, \dots, p_m < \infty$ such that there is $i \in [m]$ with $(m-1)p'_i \leq p_k$ for every $k \in [m] \setminus \{i\}$ and μ defined as in Proposition 30, then for every $\mathbf{x}, \mathbf{y} \in \mathbb{S}_{++}^{d-d_i}$ we have $\mu(G(\mathbf{x}), G(\mathbf{y})) \leq \mu(\mathbf{x}, \mathbf{y})$.

Proof. For $l \in [m] \setminus \{i\}$, let $\phi_l := \min_{j_l \in [d_l]} \frac{x_{l,j_l}}{y_{l,j_l}}$ and $\varphi_l := \max_{j_l \in [d_l]} \frac{x_{l,j_l}}{y_{l,j_l}}$. Note that $\|\mathbf{x}_l\|_{p_l} = \|\mathbf{y}_l\|_{p_l} = 1$ implies $\phi_l \leq 1 \leq \varphi_l$. By Lemmas 6 and 15, for every $l \in [m] \setminus \{i\}$ and $j_l \in [d_l]$ we have

$$\phi_l^{-1} \left(\prod_{k \in [m] \setminus \{i\}} \phi_k \right)^{p'_i} (s_{i,l,j_l}(\mathbf{y}))^{p_l-1} \leq (s_{i,l,j_l}(\mathbf{x}))^{p_l-1} \leq \varphi_l^{-1} \left(\prod_{k \in [m] \setminus \{i\}} \varphi_k \right)^{p'_i} (s_{i,l,j_l}(\mathbf{y}))^{p_l-1}.$$

It follows that for every $j_1 \in [d_1], \dots, j_m \in [d_m]$ we have

$$\prod_{l \in [m] \setminus \{i\}} \phi_l^{p_l-1} \leq \prod_{l \in [m] \setminus \{i\}} \left(\frac{s_{i,l,j_l}(\mathbf{x})}{s_{i,l,j_l}(\mathbf{y})} \right)^{p_l-1} \leq \prod_{l \in [m] \setminus \{i\}} \varphi_l^{p_l-1},$$

where we have used $\phi_l \leq 1 \leq \varphi_l$ and $1 \leq p'_i(m-1) \leq p_l$ for every $l \in [m] \setminus \{i\}$. Thus,

$$e^{\mu(G(\mathbf{x}), G(\mathbf{y}))} = \frac{\prod_{l \in [m] \setminus \{i\}} \max_{j_l \in [d_l]} \left(\frac{s_{i,l,j_l}(\mathbf{x})}{s_{i,l,j_l}(\mathbf{y})} \right)^{p_l-1}}{\prod_{l \in [m] \setminus \{i\}} \min_{j_l \in [d_l]} \left(\frac{s_{i,l,j_l}(\mathbf{x})}{s_{i,l,j_l}(\mathbf{y})} \right)^{p_l-1}} \leq \frac{\prod_{l \in [m] \setminus \{i\}} \varphi_l^{p_l-1}}{\prod_{l \in [m] \setminus \{i\}} \phi_l^{p_l-1}} = e^{\mu(\mathbf{x}, \mathbf{y})},$$

the desired inequality follows from the fact that $t \mapsto \ln(t)$ is an increasing function. \square

Now, let us prove the convergence of the sequences produced by HGPM. Note that Theorem 2 is a direct consequence of the next Theorem.

Theorem 33. Let $f \geq 0$ be weakly irreducible and $1 < p_1, \dots, p_m < \infty$ such that there is $i \in [m]$ with $(m-1)p'_i \leq p_k$ for all $k \in [m] \setminus \{i\}$ and, if $m = 2$, choose i such that $p_i \leq p_k$ for $k \in [2] \setminus \{i\}$. Let $\mathbf{x}^* \in \mathbb{S}^{d-d_i}_{++}$ be the unique strictly positive critical point of Q_i in $\mathbb{S}^{d-d_i}_{++}$. Let $(\lambda_-^k)_{k \in \mathbb{N}}, (\lambda_+^k)_{k \in \mathbb{N}} \subset \mathbb{R}$ and $(\mathbf{x}^k)_{k \in \mathbb{N}} \subset \mathbb{S}^{d-d_i}$ be the sequences produced by HGPM. Then $(\mathbf{x}^k)_{k \in \mathbb{N}}$ converges to \mathbf{x}^* , $(\lambda_-^k)_{k \in \mathbb{N}}, (\lambda_+^k)_{k \in \mathbb{N}}$ and $(Q_i(\mathbf{x}^k))_{k \in \mathbb{N}}$ converge to $\|f\|_{p_1, \dots, p_m}$ and there is a norm $\|\cdot\|_G$ on \mathfrak{R}^{d-d_i} , $0 < \nu < 1$ and $k_0 \in \mathbb{N}$ such that $\|\mathbf{x}^{k+1} - \mathbf{x}^*\|_G \leq \nu \|\mathbf{x}^k - \mathbf{x}^*\|_G$ for all $k \geq k_0$.

Proof. First, we prove that $(\mathbf{x}^k)_{k \in \mathbb{N}}$ converges to \mathbf{x}^* . Let $\|\cdot\|_G$ be the norm defined in Corollary 28 and $\mu_S : \mathfrak{R}^{d-d_i} \times \mathfrak{R}^{d-d_i} \rightarrow \mathbb{R}$ the metric defined by $\mu_S(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|_G$. Note that $\mathbb{S}^{d-d_i}_{++} \subset \mathfrak{R}^{d-d_i}$ is connected and if μ is defined as in Proposition 30, then μ is a metric on $\mathbb{S}^{d-d_i}_{++}$. Moreover, Proposition 31 implies that the topology induced by μ on $\mathbb{S}^{d-d_i}_{++}$ is the same as that inherited from $(\mathfrak{R}^{d-d_i}, \mu_S)$. By Corollary 28 we know the existence of some $r_0 > 0$ and $0 < \nu < 1$ such that for every $\mathbf{y}^0 \in \mathfrak{R}^{d-d_i}$ with $\|\mathbf{y}^0 - \mathbf{x}^*\|_G < r_0$ holds

$$\lim_{k \rightarrow \infty} \mu_S(\mathbf{y}^k, \mathbf{x}^*) = \lim_{k \rightarrow \infty} \|\mathbf{y}^k - \mathbf{x}^*\|_G \leq \lim_{k \rightarrow \infty} \nu^k \|\mathbf{y}^0 - \mathbf{x}^*\|_G = 0 \quad \text{where } \mathbf{y}^{k+1} := G(\mathbf{y}^k) \quad \forall k \in \mathbb{N}.$$

Now, let $\|\mathbf{v}\|_{\bar{p}} := \sum_{l \in [m] \setminus \{i\}} \|\mathbf{v}_l\|_{p_l}$, μ_l defined as in Equation (10) and $C > 0$ such that $\|\mathbf{v}\|_G \leq C \|\mathbf{v}\|_{\bar{p}}$ for every $\mathbf{v} \in \mathfrak{R}^{d-d_i}$. From Equation (11) we know that $\|\mathbf{x}_l - \mathbf{z}_l\|_{p_l} \leq 3 \left(e^{\frac{\mu_l(\mathbf{x}_l, \mathbf{y}_l)}{p_l-1}} - 1 \right)$ for every $\mathbf{x}_l, \mathbf{y}_l \in \mathbb{R}^{d_l}$ with $\|\mathbf{x}_l\|_{p_l} = \|\mathbf{y}_l\|_{p_l} = 1$. By Remark 22, we know that $p_k \geq 2$ for all $k \in [m] \setminus \{i\}$. It follows that for every $\mathbf{x}, \mathbf{y} \in \mathbb{S}^{d-d_i}_{++} \cap \{\mathbf{z} \in \mathfrak{R}^{d-d_i} \mid \|\mathbf{z} - \mathbf{x}^*\|_G < r_0\}$ we have

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\|_G &\leq C \|\mathbf{x} - \mathbf{y}\|_{\bar{p}} \leq C \sum_{l \in [m] \setminus \{i\}} 3 \left(e^{\frac{\mu_l(\mathbf{x}_l, \mathbf{y}_l)}{p_l-1}} - 1 \right) \leq C \sum_{l \in [m] \setminus \{i\}} 3 \left(e^{\mu_l(\mathbf{x}_l, \mathbf{y}_l)} - 1 \right) \\ &\leq 3C(m-1) \left(e^{\sum_{l \in [m] \setminus \{i\}} \mu_l(\mathbf{x}_l, \mathbf{y}_l)} - 1 \right) = 3C(m-1) (e^{\mu(\mathbf{x}, \mathbf{y})} - 1). \end{aligned}$$

In particular, the function $\varphi : \{t \in \mathbb{R} \mid t \geq 0\} \rightarrow \{t \in \mathbb{R} \mid t \geq 0\}$ defined by $\phi(t) = 3C(m-1)(e^t - 1)$ is continuous and satisfies $\varphi(0) = 0$. Finally, note that by Proposition 32 we know that $\mu(G(\mathbf{x}), G(\mathbf{y})) \leq \mu(\mathbf{x}, \mathbf{y})$ for every $\mathbf{x}, \mathbf{y} \in \mathbb{S}^{d-d_i}_{++}$. So, we may apply Lemma 29 and ensure that $\lim_{k \rightarrow \infty} \mu(\mathbf{x}^*, \mathbf{x}^k) = 0$ for every choice of $\mathbf{x}^0 \in \mathbb{S}^{d-d_i}_{++}$, i.e. $(\mathbf{x}^k)_{k \in \mathbb{N}}$ converges to $\mathbf{x}^* \in \mathbb{S}^{d-d_i}_{++}$. From Theorem 1, we know that $\|f\|_{p_1, \dots, p_m} = Q_i(\mathbf{x}^*)$ and since γ_i^-, γ_i^+ and Q_i are continuous functions on $\mathbb{S}^{d-d_i}_{++}$, we have

$$\lim_{k \rightarrow \infty} \lambda_+^k = \lim_{k \rightarrow \infty} (\gamma_i^+(\mathbf{x}^k))^{\frac{1}{p'_i(m-1)}} = (\gamma_i^+(\mathbf{x}^*))^{\frac{1}{p'_i(m-1)}} = \|f\|_{p_1, \dots, p_m} = \lim_{k \rightarrow \infty} (\gamma_i^-(\mathbf{x}^k))^{\frac{1}{p'_i(m-1)}} = \lim_{k \rightarrow \infty} \lambda_-^k,$$

and $\lim_{k \rightarrow \infty} Q_i(\mathbf{x}^k) = Q_i(\mathbf{x}^*) = \|f\|_{p_1, \dots, p_m}$. Finally, since $(\mathbf{x}^k)_{k \in \mathbb{N}}$ converges to \mathbf{x}^* , there exists $k_0 > 0$ such that $\|\mathbf{x}^k - \mathbf{x}^*\|_G < r_0$ for every $k \geq k_0$ and thus $\|\mathbf{x}^{k+1} - \mathbf{x}^*\|_G \leq \nu \|\mathbf{x}^k - \mathbf{x}^*\|_G$ for every $k \geq k_0$. \square

7 Experiments

We compare the HGPM with the Power Method (PM) proposed by Friedland et al. in [11]. The PM computes the singular values of nonnegative weakly irreducible tensors for the special case $p_1 = \dots = p_m$. The sequence $(\mathbf{v}^k)_{k \in \mathbb{N}} \subset \mathfrak{R}^d$ produced by this algorithm can be formulated as $\mathbf{w}^{k+1} = (\sigma_1(\mathbf{v}^k), \dots, \sigma_m(\mathbf{v}^k))$, $\mathbf{v}^{k+1} = \frac{\mathbf{w}^k}{\mathbf{n}^T \mathbf{w}^k}$ for $k \geq 1$ and some vector $\mathbf{n} \in \mathfrak{R}^d$ with $\mathbf{n} > 0$. For all our experiments we took $\mathbf{n} = (1, 1, \dots, 1)$ and $\mathbf{v}^0 = \frac{(1, 1, \dots, 1)}{d_1 + \dots + d_m} \in \mathfrak{R}^d$ and $\mathbf{x}^0 = \left(\frac{(1, 1, \dots, 1)}{\|(1, 1, \dots, 1)\|_{p_1}}, \dots, \frac{(1, 1, \dots, 1)}{\|(1, 1, \dots, 1)\|_{p_m}} \right) \in \mathfrak{R}^{d-d_i}$ as starting points. Plots show

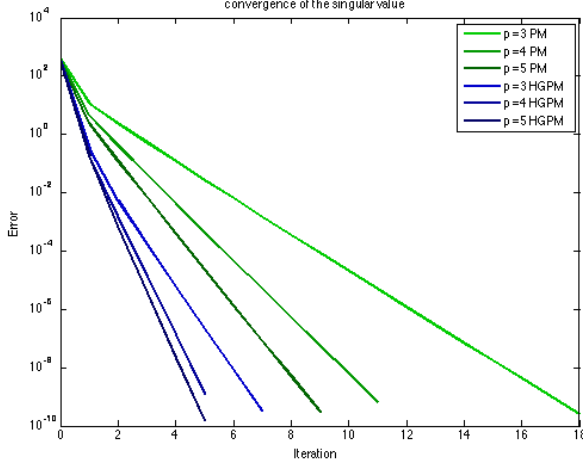


Figure 1: Plot of the error $|Q(\mathbf{v}_1^k, \dots, \mathbf{v}_m^k) - Q(\mathbf{x}^*)|$ (in green) and the error $|Q(\mathbf{x}_1^k, \dots, \mathbf{x}_{i-1}^k, \sigma_i(\mathbf{x}^k), \mathbf{x}_{i+1}^k, \dots, \mathbf{x}_m^k) - Q(\mathbf{x}^*)|$ (in blue) versus the number of iterations k on a semilogarithmic scale.

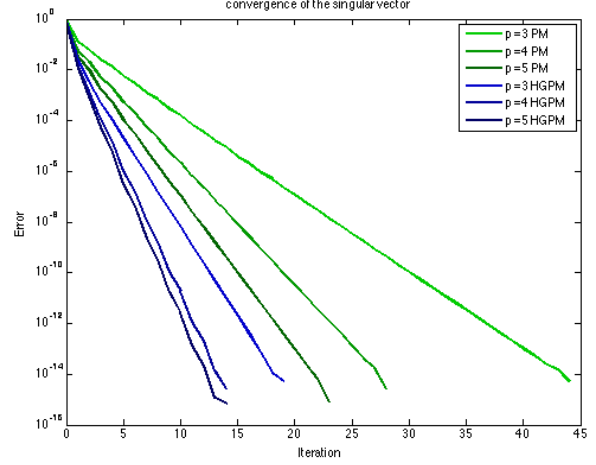


Figure 2: Plot of the error $\|\mathbf{v}^k - \mathbf{x}^*\|_2$ (in green) and the error $\|(\mathbf{x}_1^k, \dots, \mathbf{x}_{i-1}^k, \sigma_i(\mathbf{x}^k), \mathbf{x}_{i+1}^k, \dots, \mathbf{x}_m^k) - \mathbf{x}^*\|_2$ (in blue) versus the number of iterations k on a semilogarithmic scale..

linear convergence as stated in Theorem 33. Numerical experiments on randomly generated tensors showed that HGPM converges usually quicker than PM. Note also that the computation of one iteration of HGPM requires to go two times over all entries of f while PM requires it only one time. In Figure 2, we show the convergence rate of both algorithms for $p = p_1 = \dots = p_m \in \{3, 4, 5\}$ and the weakly irreducible tensor $f \in \mathbb{R}^{2 \times 3 \times 4}$ defined by

$$f_{1,2,1} = 806, f_{1,3,1} = 761, f_{1,3,4} = 3, f_{2,1,1} = 833, f_{2,2,2} = 285, f_{2,3,3} = 176 \quad \text{and} \quad f_{j_1, j_2, j_3} = 0 \quad \text{else.}$$

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References

References

- [1] L. Qi, W. Sun, Y. Wang, Numerical multilinear algebra and its applications, Frontiers of Mathematics in China 2 (2007) 501–526. doi:10.1007/s11464-007-0031-4.
- [2] G. Zhou, L. Caccetta, K. L. Teo, S.-Y. Wu, Nonnegative polynomial optimization over unit spheres and convex programming relaxations, SIAM Journal on Optimization 22 (2012) 987–1008. doi:10.1137/110827910.

- [3] P. Comon, Tensors: a brief introduction, *Signal Processing Magazine* 31 (2014) 44–53. doi:10.1109/MSP.2014.2298533.
- [4] L. K. Lim, Singular values and eigenvalues of tensors: a variational approach, in: 1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing CAMSAP'05, 2005, pp. 129–132. doi:10.1109/CAMAP.2005.1574201.
- [5] A. Defant, K. Floret, *Tensor norms and operator ideals*, Vol. 176, North-Holland Mathematics Studies, 1993.
- [6] J. M. Hendrickx, A. Olshevsky, Matrix p -norms are np-hard to approximate if $p \neq 1, 2, \infty$, *SIAM J. Matrix Anal. Appl.* 31 (2010) 2802–2812. doi:10.1137/09076773X.
- [7] C. Hillar, L.-H. Lim, Most tensor problems are np-hard, *Journal of the ACM* 60. doi:10.1145/2512329.
- [8] D. W. Boyd, The power method for ℓ^p norms, *Linear Algebra and its Applications* 9 (1974) 95–101. doi:10.1016/0024-3795(74)90029-9.
- [9] A. Bhaskara, A. Vijayaraghavan, Approximating matrix p -norms, in: *Proceedings of the Twenty-second Annual ACM-SIAM Symposium on Discrete Algorithms*, 2011, pp. 497–511.
- [10] K. C. Chang, K. Pearson, T. Zhang, Perron-Frobenius theorem for nonnegative tensors, *Communications in Mathematical Sciences* 6 (2008) 507–520. doi:10.4310/CMS.2008.v6.n2.a12.
- [11] S. Friedland, S. Gaubert, L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, *Linear Algebra and its Applications* 438 (2013) 738–749. doi:10.1016/j.laa.2011.02.042.
- [12] C. Ling, L. Qi, $l^{k,s}$ -Singular values and spectral radius of rectangular tensors, *Frontiers of Mathematics in China* 8 (2013) 63–83. doi:10.1007/s11464-012-0265-7.
- [13] M. Ng, L. Qi, G. Zhou, Finding the largest eigenvalue of a nonnegative tensor, *SIAM Journal on Matrix Analysis and Applications* 31 (2009) 1090–1099. doi:10.1137/09074838X.
- [14] K. C. Chang, K. J. Pearson, T. Zhang, Some variational principles for Z -eigenvalues of nonnegative tensors, *Linear Algebra and its Applications* 438 (2013) 4166 – 4182. doi:10.1016/j.laa.2013.02.013.
- [15] G. H. Golub, C. F. v. Loan, *Matrix computations*, 1st Edition, 1983.
- [16] S. Gaubert, J. Gunawardena, The perron-frobenius theorem for homogeneous, monotone functions, *Transactions of the American Mathematical Society* 356 (2004) 4931–4950. doi:0.1090/S0002-9947-04-03470-1.

- [17] R. D. Nussbaum, Convexity and log convexity for the spectral radius, *Linear Algebra and its Applications* 73 (1986) 59–122. doi:10.1016/0024-3795(86)90233-8.
- [18] L. Qi, Eigenvalues of a real supersymmetric tensor, *Journal of Symbolic Computation* 40 (2005) 1302–1324. doi:10.1016/j.jsc.2005.05.007.
- [19] K. C. Chang, L. Qi, G. Zhou, Singular values of a real rectangular tensor, *Journal of Mathematical Analysis and Applications* 370 (2010) 284–294. doi:10.1016/j.jmaa.2010.04.037.
- [20] R. A. Horn, C. R. Johnson (Eds.), *Matrix Analysis*, Cambridge University Press, 1986.
- [21] R. D. Nussbaum, Hilbert’s projective metric and iterated nonlinear maps, *Memoirs of the American Mathematical Society* 75 (1988) 1–137. doi:10.1090/memo/0391.